Three Ways Across a Turn: Chord, Arc and Tangent

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November 2026 (version 2.1)

Abstract. When a unit segment turns through an angle θ, its tip can be joined to where it started in three natural ways: by the straight chord, by the arc the turn actually traces, and by the two tangent lines that meet outside the circle. Their lengths are $2\sin(\theta/2)$, θ and $2\tan(\theta/2)$, and they are always in that order. All of this is classical, from Archimedes through Snellius and Huygens. This paper collects what the three paths mean when the turn is read as a complex number $U = e^{i\theta}$ or a unit quaternion. The chord is $U - 1$, and its square is Wilson's lattice energy. The arc is the logarithm, and its square is the continuum energy to leading order. The tangent leg $\tan(\theta/2)$ is the Cayley parameter that builds U algebraically, and read as a time step it is the Crank–Nicolson rule. The chord and the tangent miss the arc in opposite directions, the tangent by exactly twice as much at leading order. So the Snellius–Huygens weights 2:1 cancel the leading error both in length and in energy: $(2C^2 + T^2)/3 = \theta^2 + \frac{7}{720}\theta^6 + \dots$. Half the chord is the distance d of the identity $S + d^2 = 1$, with sameness $S = \cos^2(\theta/2)$. On the $SU(2)$ meshes of our curvature-recovery work, face angles are small enough (at most 0.094) that the chord and the arc differ in energy by under 0.08%, so these corrections are not the source of the errors measured there. Every miss has a twin on the other side, and crossing the circle's rim makes the turn imaginary while $S + d^2 = 1$ survives. Bertrand's paradox turns out to be the question of which of these quantities is taken to be uniform: the arc, $\sqrt{S}$, or S. A ticket game puts all of it in one room: the arc, the chord and the energy each buy one ticket at a different angle (57.30°, 60°, 90°), and two whole tickets can only turn at the centre or at its twin. Every identity is checked symbolically or numerically by a script that ships with the paper. In a gauge field every small loop carries a turn, its holonomy, and the picture reads directly: the turn creates the energy, and a lattice pays for it along the chord. A last section offers, as an explicitly speculative motivation and not a result, a reading of the ticket game toward Yang–Mills. A companion paper computes with these turns on the Poincaré homology sphere.

1 A picture

Draw a line of length one from the origin to the point 1. Turn it through an angle θ. Its tip moves along the unit circle from 1 to $e^{i\theta}$. Turn it in every direction and the tip traces the whole circle.

Now ask the two positions of the tip to reach each other again. There are three natural ways (Figure 1). The *chord* goes straight through the circle. The *arc* goes along the circle, the way the turn went. The *outside path* follows the tangent line at one end to the point where it meets the tangent at the other, and then comes back in. The arc is the only one of the three that the turn itself can travel; the chord is the shortcut that cuts across, and the outside path is the detour that never touches the circle in between.

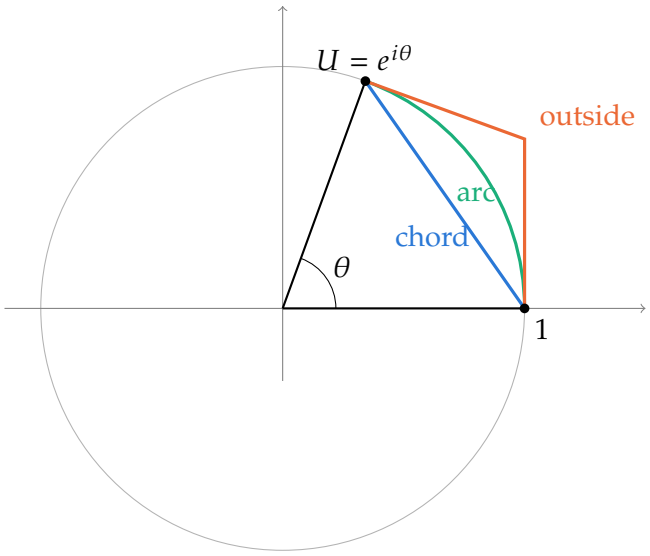

Figure 1. Three ways across a turn of $\theta = 70°$: the chord (length $2\sin(\theta/2)$), the arc (θ) and the outside path along the two tangents ($2\tan(\theta/2)$).

In plain words. Turning a stick through an angle moves its tip along a curve. Afterwards the two tip positions can be joined by cutting straight across (the chord), by going around the way the stick went (the arc), or by going out along the two edge lines and back (the outside path). The shortcut is always shortest and the detour always longest. The true path lies between them. This paper is about what each of the three means, and what it costs.

Write $C = 2\sin(\theta/2)$, $A = \theta$ and $T = 2\tan(\theta/2)$ for the three lengths. The outside corner lies on the bisector at distance $\sec(\theta/2)$ from the centre, so each tangent segment has length $\tan(\theta/2)$. Figure 2 draws the same three paths in space, on the sphere where the non-Abelian turns of Section 5 live.

What is classical, and what this paper adds. The inequality $C < A < T$ is Archimedes' squeeze of the circle between inscribed and circumscribed polygons [1]. The weighted combination of Section 2 is the Snellius–Huygens estimate [2, 3]. The chord energy is Wilson's lattice action [6], the arc energy is Manton's [7], the Cayley map is from 1846 [4], and Crank–Nicolson is from 1947 [5]. None of these identifications is new. What this paper does is read the three paths as one object: three ways of measuring the same turn, which miss in fixed proportions, with each proportion showing up again in energy, in algebra and in time stepping. It then checks how much this matters on a real non-Abelian mesh. Parts of the energy story are not new either. Cancelling the θ^4 (compactness) term of Wilson's action has been done for $U(1)$ with an adjoint, doubled-angle term [11]. For $U(1)$ and $SU(2)$ the tangent energy is, as $T^2/2$, Lüscher's topology-conserving action at its boundary value [13, 12]. The scalar inequality $\frac{2}{3}(\sin x/x)^2 + \frac{1}{3}(\tan x/x)^2 > 1$ behind Proposition 2 is a known inequality of Wilker–Huygens type [14, 15], and the chord, arc and tangent appear side by side as discrete curvatures of polygons [10]. What we have not found is the chord–tangent blend itself proposed as a lattice energy, its algebraic form in U, or its coefficients $\frac{7}{720}$ and $\frac{1}{1008}$.

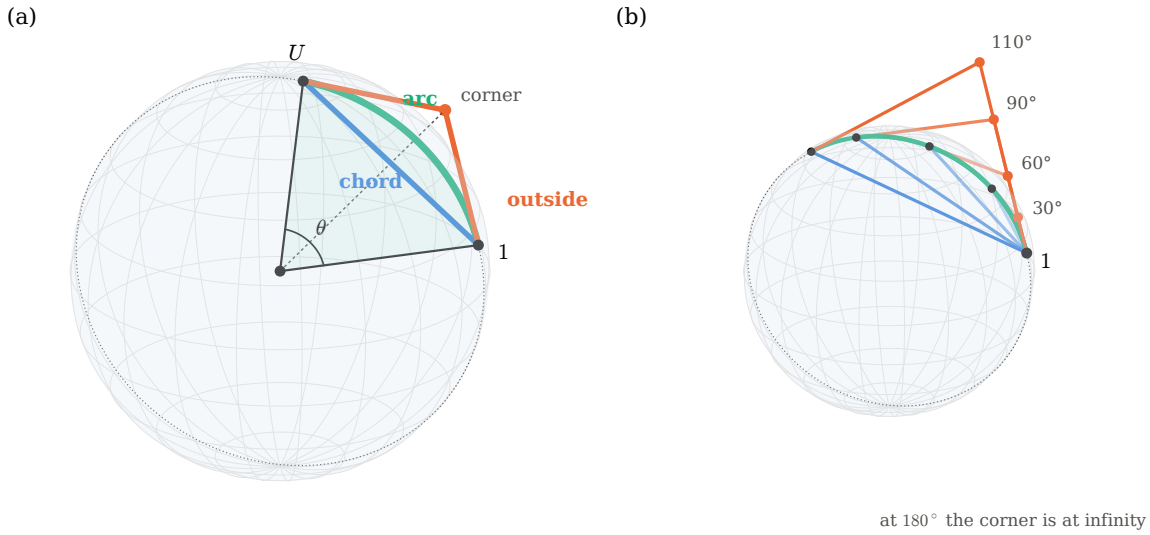

Figure 2. The three paths in three dimensions. **(a)** A turn of $\theta = 70°$ in a plane through the centre of the unit sphere: the arc (green) runs on the surface, the chord (blue) cuts through the inside, and the outside path (orange) leaves along the tangent at 1, turns at the corner on the bisector at distance $\sec(\theta/2)$, and comes back in along the tangent at U. **(b)** Turns of 30°, 60°, 90° and 110° from the same start. The chord and the arc stay bounded; the corner moves outward along the bisector and reaches infinity at the half turn, the barrier met again in Sections 4 and 5. Orthographic view; computed by `make_figures.py`.

Where this is going. The turn in this paper is not only a stick on a circle. In a gauge field every small loop carries one, its holonomy, and read that way the picture says three things. The turn creates the energy: a loop that has not turned costs nothing, and the energy of a loop is how far it has turned. The two ends of a turn reach each other through the inside: a lattice measures the chord, because the arc is the path that only the turning itself can travel. And the smallest unit of turn is curved because it exists: one unit, extending equally in every direction, is a sphere, and in four dimensions that sphere is $SU(2)$, the group the Yang–Mills field is made of. The companion paper *Reading the Curve, Not Counting the Points* [24] computes with these turns on meshes of the Poincaré homology sphere. Section 9 measures how much the three paths differ there, and Section 11 returns to what they suggest about Yang–Mills. Section 12 says how far this paper goes, and where it stops.

2 Lengths: who misses, and by how much

Proposition 1. *For $0 < \theta < \pi$, $C < A < T$, and*

$$A - C = \tfrac{1}{24}\theta^3 + O(\theta^5), \qquad T - A = \tfrac{1}{12}\theta^3 + O(\theta^5).$$

The outside path overshoots by exactly twice what the chord falls short, at leading order. Hence

$$\frac{2C + T}{3} = \theta + \tfrac{1}{320}\theta^5 + O(\theta^7),$$

and $(2C + T)/3 > \theta$ on the whole interval (the Snellius–Huygens inequality $2\sin x + \tan x > 3x$ for $0 < x < \pi/2$, with $x = \theta/2$).

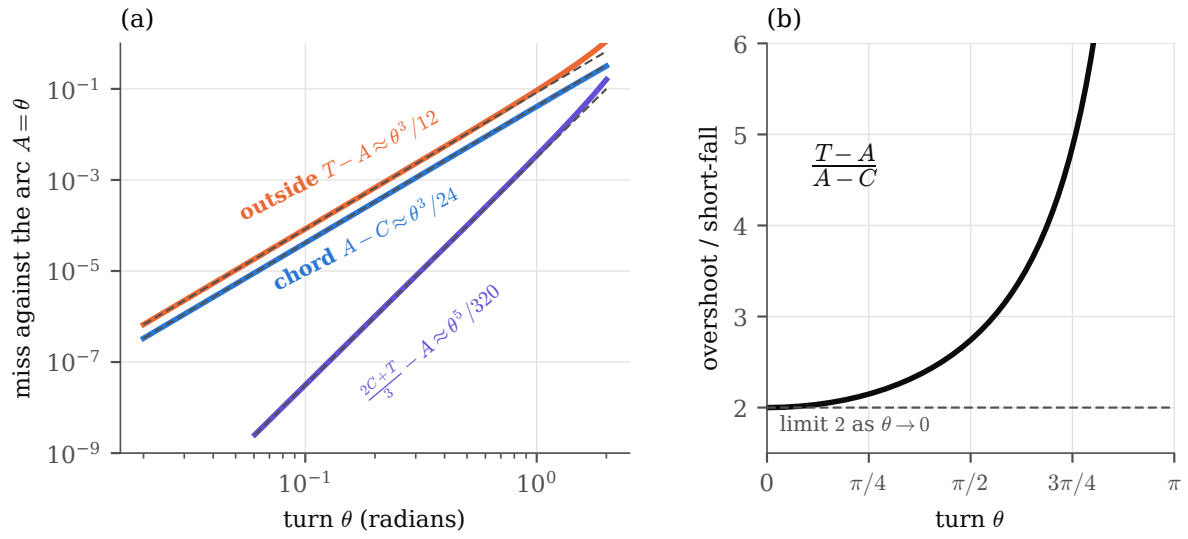

Figure 3. Lengths. **(a)** The chord's short-fall $A - C$ and the outside path's overshoot $T - A$ both fall like θ^3, at $\theta^3/24$ and $\theta^3/12$ (dashed: the leading terms of Proposition 1); the 2:1 Snellius–Huygens blend falls like $\theta^5/320$. **(b)** The ratio of overshoot to short-fall is 2 in the limit and grows with the turn; the cancellation is a small-angle statement.

Proof. The series follow from $\sin x = x - x^3/6 + x^5/120 - \dots$ and $\tan x = x + x^3/3 + 2x^5/15 + \dots$ at $x = \theta/2$. The inequalities are classical [1, 3]; all are checked by `verify.py`. □

In plain words. The shortcut falls short of the true path, and the detour overshoots it. For small turns the overshoot is always exactly twice the shortfall. So if you take the shortcut twice and the detour once and average, the two errors cancel, and you get the true length almost perfectly: the leftover error shrinks like the fifth power of the angle. Snellius found this in 1621 and Huygens proved it in 1654. Neither side alone tells the truth; the two sides together nearly do.

3 The turn as a number

Read the turn as the complex number $U = e^{i\theta}$. Then the chord is the complex number $U - 1$, and

$$U - 1 = 2i \sin(\theta/2)\, e^{i\theta/2}.$$

Its length is C, and its direction is the starting direction turned by $\theta/2$ (plus a quarter turn). *The chord turns at half the speed of the line.* For $0 < \theta < 2\pi$ the chord is nonzero and its direction is $\theta/2 + \pi/2$; as the line turns once around, $\theta \to 2\pi$, that direction moves only half way. At $\theta = 2\pi$ itself the chord $U - 1$ is zero and has no direction, so this is a statement about the open interval, followed continuously, not by itself a double-cover construction.

The chord and sameness. Write

$$S = \cos^2(\theta/2) = \tfrac{1}{2}(1 + \operatorname{Re} U), \qquad d = \sin(\theta/2) = \tfrac{1}{2}C.$$

Then $S + d^2 = 1$, and $1 - S = C^2/4$. This is the identity of the double-cover paper [25], with d read as half the chord and S as the sameness of the two positions. Two positions are fully the same ($S = 1$) when the chord vanishes. They are fully different ($S = 0$) at the half turn, where the chord is a diameter.

The tangent and the Cayley map. The tangent leg $\tau = \tan(\theta/2)$ builds the turn algebraically, with no exponential:

$$U = \frac{1 + i\tau}{1 - i\tau}, \qquad \tau = \frac{\sin\theta}{1 + \cos\theta} = \frac{\operatorname{Im} U}{1 + \operatorname{Re} U} = \frac{\operatorname{Im} U}{2S}.$$

This is the Cayley transform [4]. As $\theta \to \pi$, $\tau \to \infty$: at the half turn the two tangents are parallel, never meet, and the outside path has no corner. The Cayley map reaches every turn except the half turn.

In plain words. As a number, the turn is a point on the unit circle. The shortcut is "where you are minus where you started". Its direction turns half as fast as the stick does, which is why one full turn of the stick only half-turns the shortcut. Half the shortcut's length is exactly the distance d in the double-cover identity $S + d^2 = 1$, so the shortcut measures how different the two positions are. The detour's leg is the one number that rebuilds the whole turn with only arithmetic, no exponentials. At a half turn it breaks: the two edge lines are parallel and never meet.

4 Energy

In lattice gauge theory the turn U is the holonomy around a face, and the face's energy measures how far U is from 1 [6]. Each of the three paths gives an energy:

$$C^2 = 2(1 - \cos\theta) = \theta^2 - \tfrac{1}{12}\theta^4 + \tfrac{1}{360}\theta^6 - \dots, \qquad A^2 = \theta^2, \qquad T^2 = \theta^2 + \tfrac{1}{6}\theta^4 + \tfrac{17}{720}\theta^6 + \dots$$

C^2 is Wilson's action (up to normalisation), and A^2 is Manton's geodesic action [7], which is the continuum energy of the face to leading order in its size (the holonomy alone does not fix the curvature inside a face). The tangent energy again misses on the other side, and again by exactly twice as much at leading order. So the same Snellius–Huygens weights cancel it.

Proposition 2.

$$\frac{2C^2 + T^2}{3} = \theta^2 + \tfrac{7}{720}\theta^6 + \tfrac{1}{1008}\theta^8 + O(\theta^{10}).$$

The θ^4 term cancels exactly. The combination is algebraic in U: $C^2 = |U - 1|^2$ and $T^2 = 4\,|\operatorname{Im} U|^2/(1 + \operatorname{Re} U)^2$.

In plain words. The energy of a turn can be measured with the shortcut, the true path or the detour. Lattice physics has used the shortcut since 1974 because it is easy to compute; the true path is what the continuum means. The shortcut energy is slightly too small and the detour energy slightly too big, by twice as much. Mixing them two-to-one cancels the main error, using only arithmetic on U.

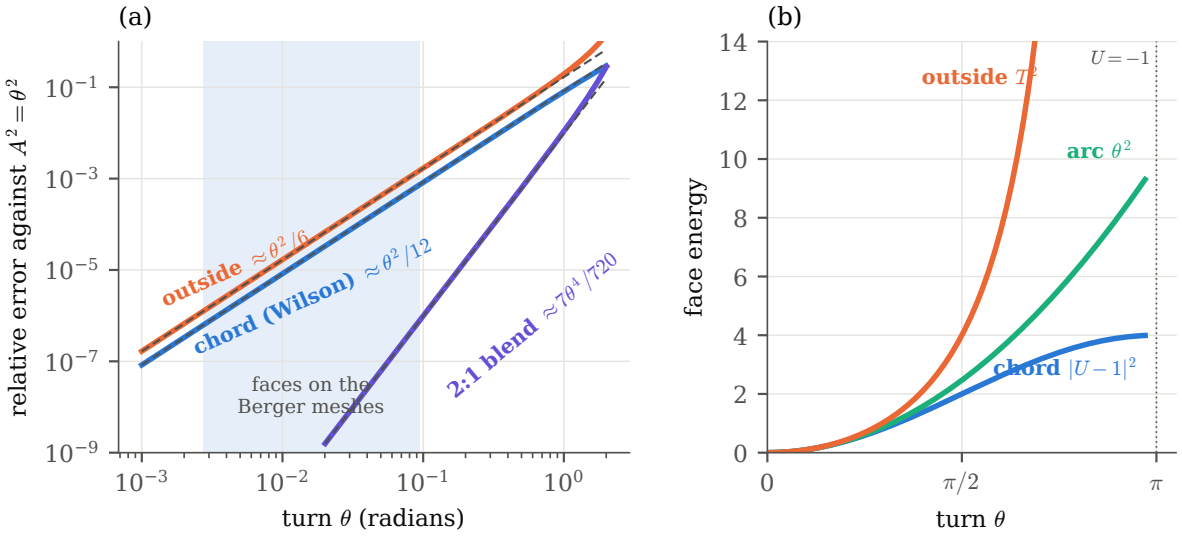

FIGURE 4. Energy. **(a)** Relative error of each face energy against the continuum energy θ^2: the chord (Wilson's action) at $\theta^2/12$, the outside path at $\theta^2/6$, and the 2:1 blend at $7\theta^4/720$ (Proposition 2; dashed lines are the leading terms). The shaded band is the range of face angles on the $SU(2)$ Berger meshes of Section 9, from the median at $n = 5$ to the largest at $n = 1$. **(b)** The three energies over the whole turn: the chord's is bounded by 4, the arc's is θ^2, and the outside path's has a barrier at $U = -1$.

Remark 1 (What this does and does not correct)**.** *On a lattice of spacing a and a smooth field, $\theta = O(a^2)$, so the θ^4 term is a relative correction of order a^4. The dominant lattice error is of order a^2 and comes from the shape of the face, not from the compactness of the group; it is what Symanzik improvement removes [8]. The combination above addresses only the compactness error. That error matters for strong fields and coarse lattices, and is negligible for smooth fields on fine ones. The tangent energy also diverges at the half turn $U = -1$, unlike Wilson's and like the Cayley map. Lattice actions with exactly this barrier have been studied as topology-conserving actions [13, 12]: they freeze topology in Monte Carlo, and Creutz has argued that such admissibility conditions are incompatible with a positive transfer matrix [9]. For the blend, the barrier's effect in a simulation is not studied here.*

5 The non-Abelian turn

A unit quaternion (an element of $SU(2)$) is $U = \cos a + n \sin a$ with n a pure unit quaternion. Everything above carries over with θ replaced by the geodesic angle a on S^3:

$$|U-1| = 2\sin(a/2), \qquad \log U = a\,n, \qquad U = (1+v)(1-v)^{-1} \text{ with } v = \frac{\operatorname{Im} U}{1+\operatorname{Re} U},\ |v| = \tan(a/2).$$

The quaternion U rotates three-space by $2a$, so the chord, arc and outside path of the quaternion live on the double cover of the rotation they describe. The Cayley vector v again reaches everything except $U = -1$, the other sheet's copy of the identity. `verify.py` checks all three on 2,000 random quaternions. The computed chord and arc agree with their closed-form values to 10^{-14}. The Cayley reconstruction agrees to 10^{-10}; its conditioning grows like $1/(1+\cos a)$ near the half turn.

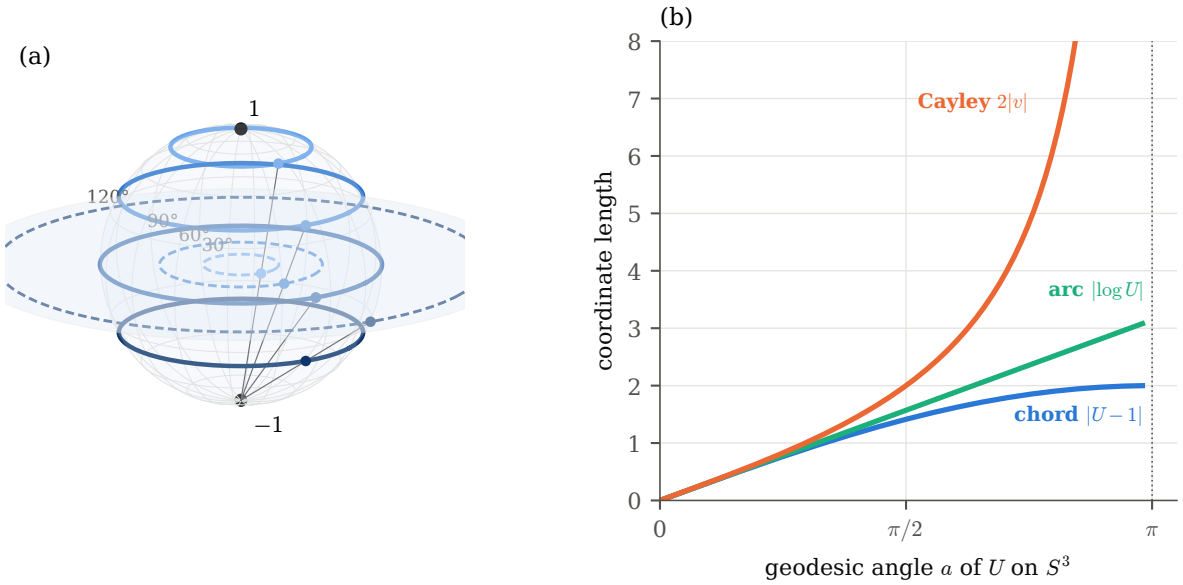

FIGURE 5. The Cayley vector as stereographic projection, drawn on S^2 in place of S^3. **(a)** The vertical axis is $\operatorname{Re} U$. Solid circles are the turns at geodesic angle $a = 30°, 60°, 90°, 120°$ from 1; dashed circles, in the same shade, are their Cayley images in the plane $\operatorname{Re} U = 0$, of radius $\tan(a/2)$. The thin rays run from the projection point -1. As $a \to \pi$ the image runs to infinity, which is why the Cayley map reaches everything except $U = -1$. **(b)** The three coordinates of one turn: chord $|U - 1| = 2\sin(a/2)$, arc $|\log U| = a$, and Cayley $2|v| = 2\tan(a/2)$. They are the lengths C, A and T of Section 2, read on S^3.

IN PLAIN WORDS. The same three paths exist for turns in three dimensions, written as quaternions. The formulas are identical with the turning angle measured on the three-sphere. The detour still rebuilds the turn with plain arithmetic, and it still breaks only at the half turn.

The Cayley vector has a picture. Since $v = \operatorname{Im} U/(1 + \operatorname{Re} U)$, it is the stereographic projection of U from the point -1 onto the equatorial space $\operatorname{Re} U = 0$. Figure 5 draws it one dimension down, on S^2.

6 TIME STEPPING

Take the simplest motion that turns: $y' = i\,y$, whose exact solution goes around the circle. One step of size s should turn y by s. The three classical one-step rules put the point in three places, and they are the three paths again.

- **Explicit Euler**, $y \mapsto (1 + is)\,y$, steps out along the tangent line. Its modulus is $\sqrt{1 + s^2} > 1$, so it leaves the circle outward and gains energy every step.
- **Implicit Euler**, $y \mapsto y/(1 - is)$, has modulus $1/\sqrt{1 + s^2} < 1$. It falls inside the circle and loses energy every step.
- **Crank–Nicolson**, $y \mapsto \frac{1+is/2}{1-is/2}\,y$, is the Cayley map with $\tau = s/2$. It stays exactly on the circle, but turns by $\varphi = 2\arctan(s/2)$. The step s plays the role of the outside path T, and the turn actually made is the arc. The phase lag is $s - \varphi = \frac{1}{12}s^3 + O(s^5)$, the same $1/12$ as $T - A$ in Proposition 1.

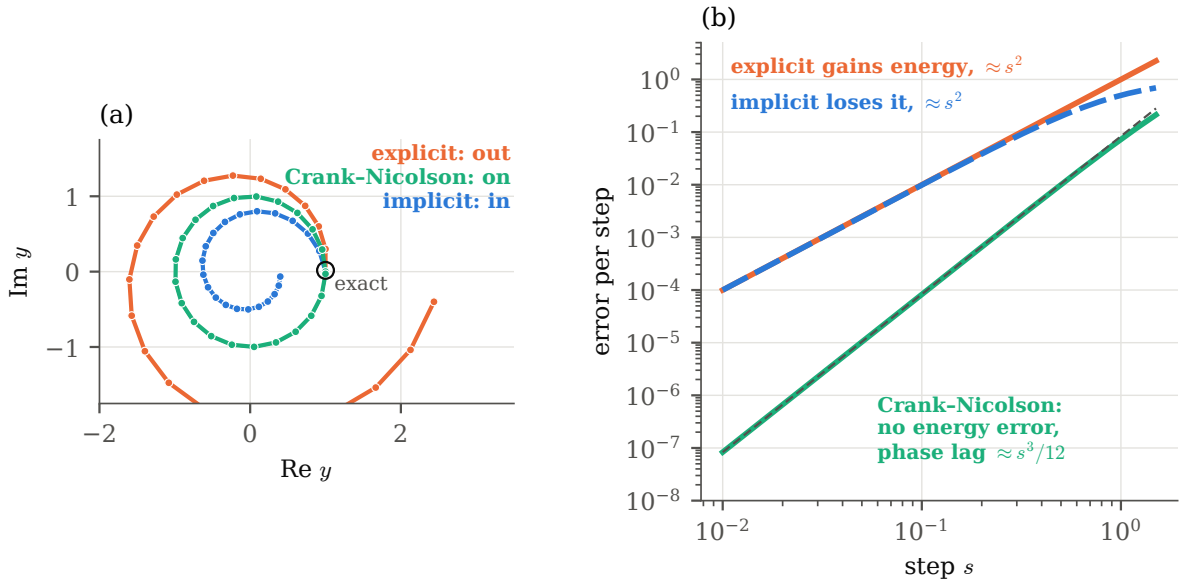

FIGURE 6. Time stepping on $y' = iy$. **(a)** Twenty-one steps of size $s = 0.3$ (one full turn, $t \approx 2\pi$) from $y = 1$. Explicit Euler (orange) spirals out, implicit Euler (blue) spirals in, and Crank–Nicolson (green) stays on the circle; the open circle marks the exact endpoint. **(b)** Error per step: the Euler rules gain or lose energy at order s^2 (the explicit and implicit curves agree at leading order), while Crank–Nicolson makes no energy error and lags in phase by $s - 2\arctan(s/2) \approx s^3/12$ (dashed).

Figure 6 shows the three rules over one full turn.

THREE SPIRALS, ONE OF THEM GOLDEN. Each explicit Euler step multiplies the modulus by $\sqrt{1+s^2}$ and turns by $\arctan s$, so the ratio of growth to turning never changes. Every point it produces therefore lies exactly on a logarithmic (equiangular) spiral, Jacob Bernoulli's *spira mirabilis*,

$$r = e^{b\theta}, \qquad b(s) = \frac{\ln(1+s^2)}{2\arctan s}.$$

Implicit Euler lies on the mirror spiral $r = e^{-b\theta}$, its twin at the same pitch, and Crank–Nicolson is the member of pitch zero: the circle between them. At the step of Figure 6(a), $s = 0.3$, the outward spiral grows by a factor 1.2614 every quarter turn, at a pitch of 8.41°. The golden spiral is the member of this family that grows by $\varphi = (1+\sqrt{5})/2$ every quarter turn, $b = 2\ln\varphi/\pi \approx 0.3063$, at a pitch of 17.03°. Fibonacci's spiral of quarter circles drawn in squares approximates it, because the ratios of consecutive Fibonacci numbers tend to φ. Since $b(s)$ increases with s, there is exactly one step at which explicit Euler traces the golden spiral exactly: $s^* \approx 0.6515$, or 33.08° of turn per step. The golden case belongs to that one step size. It is a member of the family the method draws, not a property of the method.

For a general linear flow $y' = -Gy$, the Crank–Nicolson factor $r(z) = (1 - z/2)/(1 + z/2)$ satisfies $|r(z)| \le 1$ exactly when $\operatorname{Re} z \ge 0$. Its behaviour on a non-normal G is controlled by the numerical range $W(G)$. By Crouzeix–Palencia [21], $\|r(hG)^k\| \le (1+\sqrt{2})\sup_{z\in W(G)} |r(hz)|^k$. On a Lie group, the Cayley map keeps each update exactly on $SU(2)$ [16].

In plain words. Simulating something that spins means stepping it forward a little at a time. The plainest step walks out along the edge line and drifts outward, gaining energy it should not have. The backward version drifts inward and loses energy. The Cayley step, the detour's leg, stays exactly on the circle and only lags a little behind, by the same one-twelfth that separates the detour from the true path. Drawn over many steps, the outward drift traces a logarithmic spiral, the same family as the golden spiral, and at one step size it is exactly golden. The inward drift is its mirror twin, and the Cayley step is the circle between them. That made it the natural rule for simulating the flow on the mesh, and the companion paper now uses it: eight Cayley steps follow the flow as closely as many small explicit ones. The engine's own clock still steps along the true curve (Section 10).

7 Twins, the far side, and Bertrand

Every miss has a twin on the other side. The Snellius–Huygens estimate $(2C + T)/3$ overshoots the arc. Snellius' other estimate undershoots it:

$$L = \frac{6\sin(\theta/2)}{2 + \cos(\theta/2)}, \qquad \theta - L = \tfrac{1}{2880}\theta^5 + O(\theta^7), \qquad L < \theta < \frac{2C + T}{3}.$$

The two misses are $\theta^5/320$ above and $\theta^5/2880$ below, nine to one. So the twins can be blended again, $\frac{1}{10}\big(9L + \frac{2C+T}{3}\big) = \theta + O(\theta^7)$, and the blend leaves a smaller pair of twins. Each cancellation produces the next.

The far-side twin. A chord does not know which way round the turn went. The turns θ and $2\pi - \theta$ have the same chord. The outside path does know: its tangent leg changes sign, $\tan\big((2\pi - \theta)/2\big) = -\tan(\theta/2)$, because the corner where the tangents meet moves to the other side of the circle.

Where the corner sits. The chord's line lies at distance $p = \cos(\theta/2)$ from the centre, and the corner at distance $1/p$ (pole and polar). The corner's position decides the regime.

- *Corner outside the circle* ($0 < p < 1$): everything is real.
- *Chord line through the centre* ($p = 0$, the half turn): the tangents are parallel and the corner is at infinity. Past it, the corner comes back on the far side; that is the far-side twin. Crossing the centre flips to the twin, and everything stays real.
- *Corner inside the circle* ($p > 1$; the chord's line misses the circle): the two endpoints become a complex-conjugate pair, and the turn becomes imaginary, $\theta = i\varphi$. Then $C = 2i\sinh(\varphi/2)$, the tangent leg is $i\tanh(\varphi/2)$, and $S = \cosh^2(\varphi/2) > 1$. The identity $S + d^2 = 1$ survives, with $d = i\sinh(\varphi/2)$ imaginary. The circle has become a hyperbola, and the turn a boost. Crossing the rim is what makes the numbers unreal.

Bertrand's paradox. Bertrand asked for the probability that a "random chord" is longer than the side of the inscribed equilateral triangle [22]. The answer depends on how the chord is chosen, and each classical choice makes one of this paper's quantities uniform:

Chord chosen by	Uniform quantity	Probability
two random endpoints	the arc θ	1/3
random distance along a radius	$\sqrt{S} = \cos(\theta/2)$	1/2
random midpoint	the sameness S	1/4

The chord exceeds $\sqrt{3}$ exactly when $\theta > 2\pi/3$, $\cos(\theta/2) < 1/2$ and $S < 1/4$, so the three answers follow at once. A simulation with two million chords per method gives 0.3333, 0.5004 and 0.2498. The outside path offers no fourth answer of this kind: its leg runs off to infinity at the half turn, so it cannot be spread evenly. Jaynes argued that invariance under moving and rescaling the circle selects the radius method [23].

A group of turns does give a fourth answer, because it comes with its own fair way to choose a turn at random, the Haar measure, which leaves no choice to us. For a uniformly random unit quaternion the geodesic angle a has density $(2/\pi)\sin^2 a$ on $[0, \pi]$, so sameness $S = \cos^2(a/2)$ follows the semicircle law $(8/\pi)\sqrt{S(1-S)}$ on $[0, 1]$. Bertrand's event is $S < 1/4$, and its probability is

$$\tfrac{1}{3} - \tfrac{\sqrt{3}}{4\pi} \approx 0.19550,$$

none of the three classical answers. A simulation of four million random turns gives 0.19545 (`scripts/haar_sameness.py` in the GIGI repository).

> In plain words. Every estimate that misses on one side has a twin that misses on the other, and blending the twins cancels the miss, leaving a smaller pair. A chord also has a twin turn: the long way around gives the same chord, but the detour notices, because its corner jumps to the other side. Where that corner sits decides everything. Outside the circle, all is real. Through the centre, you flip to the twin. Inside the circle, the numbers become imaginary, the circle becomes a hyperbola, and still $S + d^2 = 1$ holds. Bertrand's famous paradox is the same question in disguise: "random chord" means nothing until you say which of these quantities you are spreading evenly, the arc, the square root of sameness, or sameness itself.

8 The ticket game

The preceding sections each look at one aspect of a turn. This section puts them in one room. Everything in it is elementary, and every number is checked by `verify.py`. The point is not new mathematics; it is that the three paths, the energy, the twins and Bertrand's triangle all appear the moment one asks a plain question about walking.

The rules. Everyone starts at the centre O of a room. Each person is handed tickets, and a ticket buys one straight walk of length at most 1. Two players, A and B, each spend one ticket walking out from O in a direction of their own choosing, so they end on the unit circle, an angle θ apart. Now A wants to rejoin B. One rule is never relaxed: no single walk may cost more than one ticket.

Three prices for one crossing. The crossing has three natural prices, one for each way of measuring it.

- *The arc*, the way the turn went, costs θ tickets. It fits in one ticket up to exactly one radian, 57.30°. That is what a radian is: the angle whose arc is one ticket long.
- *The chord*, straight across, costs $2\sin(\theta/2)$. It fits in one ticket up to exactly 60°. There A, B and O form an equilateral triangle, and six players spaced 60° apart stand at the corners of a regular hexagon whose sides equal its radius. That hexagon is where Archimedes began.

- *The energy*, as the lattice counts it, is Wilson's face energy $1-\cos\theta = \frac{1}{2}C^2$. In these units a single walk of length L costs $\frac{1}{2}L^2$, so the two radius walks back through the centre cost $\frac{1}{2}+\frac{1}{2}=1$ together. Call that one ticket of energy. The crossing costs exactly one ticket of energy at 90°, less below and more above, up to two at the opposite side of the room.

Each price has its own one-ticket angle, and the three are different.

Price	Cost of the crossing	One ticket at	Why
arc	θ	57.30°	the definition of a radian
chord	$2\sin(\theta/2)$	60°	the equilateral triangle
energy	$1-\cos\theta$	90°	Pythagoras

The full grid. The table above prices the crossing three ways, but each path has both a length and an energy, so the board has six squares, and each is forced once its rule is fixed. With energy measured as $\frac{1}{2}L^2$, as above:

One ticket of…	outside path	arc	chord
length	53.13° the 3–4–5 triangle	57.30° one radian	60° the hexagon
energy	70.53° $\cos\theta=\frac{1}{3}$	81.03° $\sqrt{2}$ radians	90° the square

In each row the outside path, the dearest, buys the least turn, and the chord, the cheapest, buys the most. Two squares are classical angles in their own right. The outside path buys one ticket of length when its leg is $\tan(\theta/2)=\frac{1}{2}$, and the Cayley construction with that leg gives the point $(\frac{3}{5},\frac{4}{5})$. With any rational leg t the construction gives the rational point $\big((1-t^2)/(1+t^2),\ 2t/(1+t^2)\big)$, which is how every Pythagorean triple arises. The outside path buys one ticket of energy at $\cos\theta=\frac{1}{3}$, the dihedral angle of the regular tetrahedron.

The sliver. Between 57.30° and 60° the shortcut fits in one ticket and the true path does not. The size of that gap at the hexagon angle is $\pi/3-1\approx 0.0472$ tickets, a premium of 4.72%, and it is the same number as the ratio of the circle's circumference to the hexagon's perimeter, $2\pi/6$, minus one. This is the price of curvature at the scale of one ticket. It is what Section 2 computes, $A-C\approx\theta^3/24$, seen at a size one can walk.

Why 90° is where the energy breaks even. The squared crossing is $|A-B|^2=|A|^2+|B|^2-2A\cdot B=2-2\cos\theta$. The route back through the centre has two legs of length one. In energy, the crossing and the two legs cost the same exactly when $A\cdot B=0$: at a right angle, where the chord is the hypotenuse of the two tickets. Below 90°, cutting across is an energy bargain. Above 90°, two separate trips through the centre cost less energy than the single crossing. Energy grows with the square of length, so splitting a long walk into two short ones saves energy, and past the right angle the split wins.

Whole tickets: why it is always two trips. Now tighten the rule: a ticket must be spent in full, so every walk has length exactly one. With one whole ticket, A can only land on the circle of radius one around A. Unless B stands exactly there (at $\theta=60°$), A overshoots or falls short. With two whole tickets, A can reach any point within distance two, but the corner where A turns must lie one ticket from A and one ticket from B. For $0<\theta<\pi$, two circles of radius one around

A and B meet in exactly two points, symmetric across the chord (at $\theta = \pi$ they touch only at O; at $\theta = 0$ they coincide). One is the centre O; the other is its reflection through the chord's midpoint, the point $A + B$. So there are exactly two two-ticket routes:

- *through the centre*: walk home, then out along B's path;
- *through the twin corner* $A + B$, the corner opposite O in the rhombus O, A, $A + B$, B, whose four sides all have length one.

The two corners are twins in the sense of Section 7: two answers to one question, mirror images across the chord. At the half turn they merge, since $A + B = O$ when $B = -A$, and only the route through the centre is left.

Where the twin corner stands. The twin corner is at distance $|A + B| = 2\cos(\theta/2)$ from the centre, twice the distance of the chord's midpoint. It is outside the circle below 120°, inside beyond, and exactly on the circle at 120°. At that angle the crossing is $\sqrt{3}$, the side of Bertrand's triangle, and the twin corner lands on the hexagon vertex between the two players: the rhombus splits into two equilateral triangles.

Where the detour's corner stands. The outside path of Section 1 turns at the tangent corner Q, at distance $\sec(\theta/2)$ from the centre on the same bisector. The tangent corner and the twin corner coincide exactly when $\sec(\theta/2) = 2\cos(\theta/2)$, that is, at $\theta = 90°$. There the rhombus is a square, its far corner is where the tangents meet, each tangent leg is $\tan 45° = 1$, and the outside path is exactly two whole tickets. Below 90° the tangent corner is the nearer of the two and the detour costs less than two tickets; above 90° it is the farther and the detour costs more.

The twin of the right angle. The energy of crossing to B is $1 - \cos\theta$; the energy of crossing to the point opposite B is $1 + \cos\theta$. They always sum to two tickets, because A, B and $-B$ make a right angle at A (Thales). In the notation of Section 3 the two energies are exactly $2d^2$ and $2S$, so this is $S + d^2 = 1$ once more, and the break-even at 90° is the point where $S = d^2 = \frac{1}{2}$: the two positions are exactly as much the same as they are different.

Which energy rule. The right angle is the break-even for Wilson's rule, energy $\frac{1}{2}C^2$. It is a rule, not a fact about the room. With the continuum rule, energy $\frac{1}{2}\theta^2$, the break-even moves to 81.03°; with the outside path's rule, $\frac{1}{2}T^2$, to 70.53°. What decides among them is the small-ticket limit, where the rules must agree with the continuum: at 5° the three energies are 0.003805, 0.003808 and 0.003813, within 0.2%, and they become identical as the tickets shrink. Large turns are where lattice rules differ, and Section 4 is about exactly that difference.

So the right angle is special three times over. Under Wilson's rule the crossing costs one ticket of energy, and its energy equals the two trips through the centre. With no rule at all, by geometry alone, the detour becomes two whole tickets, through the corner of a square.

In plain words. Picture a room. Everyone starts in the middle with tickets, and one ticket pays for one straight walk of at most one step. Two friends each spend a ticket walking out in different directions. How does one get back to the other? Walking the curve costs one ticket only if they are at most about 57° apart. Cutting straight across costs one ticket if they are at most 60° apart. Counting the way physicists count energy, the straight crossing costs one ticket at exactly 90°, the same as walking back to the middle and out again; that is Pythagoras. If tickets must be spent whole, then for any two points short of opposite there are exactly two ways to do it in two trips: turn at the middle of the room, or turn at its mirror-image corner on the far side of the crossing.

Table 1. The ticket game at five angles: the three prices of the crossing, its Wilson energy, and the distances from the centre of the twin corner $A + B$ and the tangent corner Q.

θ	arc	chord	outside	energy	twin corner	tangent corner
57.30°	1.0000	0.9589	1.0926	0.4597	1.7552	1.1395
60°	1.0472	1.0000	1.1547	0.5000	1.7321	1.1547
90°	1.5708	1.4142	2.0000	1.0000	1.4142	1.4142
120°	2.0944	1.7321	3.4641	1.5000	1.0000	2.0000
180°	3.1416	2.0000	∞	2.0000	0	∞

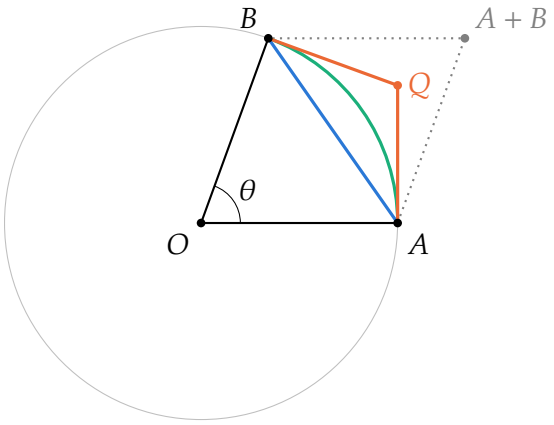

Figure 7. The ticket game at $\theta = 70°$. Each player walked one ticket from the centre. The chord (blue), the arc (green) and the outside path through the tangent corner Q (orange) are the three prices of the crossing. The two whole-ticket routes turn at the centre or at its twin $A + B$, the far corner of the unit rhombus (dotted). At 90° the tangent corner and the twin corner coincide.

Those two corners are twins. At 120° the twin corner lands on the circle itself, and at 90° it becomes the corner of a perfect square, where the two edge lines meet and the outside path is exactly two whole tickets.

9 How much this matters on a real mesh

Our curvature-recovery work [24] measures an $SU(2)$ force on meshes of the Poincaré homology sphere, at the Levi-Civita connection of a Berger metric. Its native action is the chord energy: each face's defect is $D = U_f - 1$. Table 2 gives the face angles a on those meshes (`face_angles.py`, from the frozen inputs), and Figure 8 shows their distributions.

The angles shrink roughly like the face area, and the chord–arc energy gap is below 0.08% everywhere. A small energy gap does not by itself bound the gap between forces, which are derivatives: for one face, differentiating $2(1 - \cos a)$ and a^2 gives the ratio $\sin a/a$, not the energy ratio. So the force was measured directly. That work reads the arc, through $\log U$, in its recovery tests and the chord, $U - 1$, throughout its force; with the arc in the force's recovery instead, its $SU(2)$ force changes by 0.090%, 0.0084% and 0.0021% of its norm at $n = 1, 2, 3$, and its error (between 4.7% and 36%) by at most 0.011 percentage points. The chord-versus-arc choice is not what limits that work. Where the three paths do matter for that programme is in time: its finest

Table 2. Face holonomy angles on the $SU(2)$ Berger meshes, and the relative gap between chord and arc energy at the largest angle, $a^2/12$.

n	faces	median a	max a	chord–arc energy gap at max
1	480	0.0684	0.0937	7.3×10^{-4}
2	3,840	0.0172	0.0250	5.2×10^{-5}
3	12,960	0.0076	0.0113	1.1×10^{-5}
5	60,000	0.0027	0.0041	1.4×10^{-6}

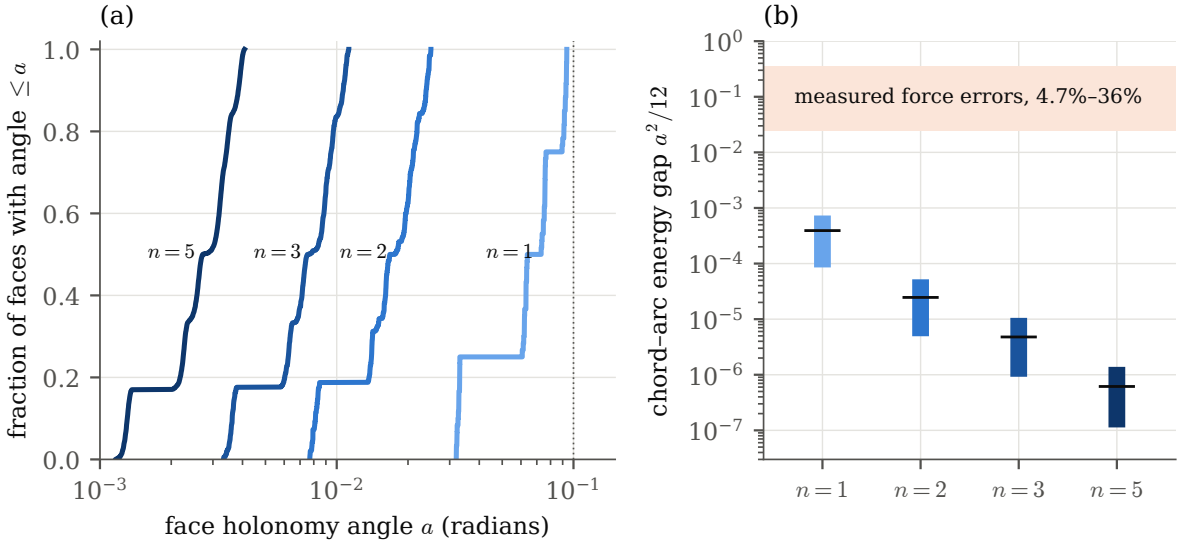

Figure 8. Face angles on the $SU(2)$ Berger meshes, from the frozen inputs (the numbers of Table 2; `make_figures.py` asserts that they agree). **(a)** The cumulative distribution of face holonomy angles at each level; every face turns by less than 0.1 radian. **(b)** The chord–arc energy gap $a^2/12$ per face: the bar spans the smallest to the largest face and the tick marks the median. The shaded band is the range of force errors measured on the same meshes. The gap lies two to five orders of magnitude below it.

flow level is stepped with the Cayley rule of Section 6.

In plain words. On the meshes we actually use, every face turns by less than a tenth of a radian. There the shortcut and the true path differ in energy by less than a tenth of a percent, far smaller than the errors we measure. So this idea does not explain those errors, and we say so. Where it does matter is in stepping the motion forward in time, and that work now uses the Cayley step to do it.

10 The computation in GIGI

The numbers in this paper sit in two places, and it matters which is which. Some of the three paths are already part of GIGI, the open-source geometric engine our curvature-recovery work runs in (PolyForm Noncommercial 1.0.0, `https://github.com/nurdymuny/gigi`). Others exist so far only in this paper's checking script. Table 3 says which. Paths are relative to the repository root at commit `442c13d` of the `feat/reading-the-curve` branch.

Table 3. Where each idea of the paper lives. "Engine" means native Rust in gigi; "script" means Python in this directory or the repository's `scripts/`.

Idea	Where	Kind
The turn as a number, $q_0 = \cos a = \frac{1}{2}\operatorname{Re}\operatorname{tr} U$	`PLAQUETTE` primitive, `src/gauge/plaquette.rs`, one q_0 per face	engine
Half the chord and sameness, $1 - S = d^2 = (1 - \cos\theta)/2$	`chord_d_sq`, `src/dials.rs`	engine
Chord energy, $D = U_f - 1$, weighted by the face mass	`cell_action`, `src/gauge/ricci_simplicial.rs`	engine
Exact links (transport along the arc) on the Berger metric	`levi_civita_link`, `src/gauge/ricci_link.rs`	engine
The arc as a map, exp on $\mathfrak{su}(2)$	Rodrigues exponential, `src/gauge/lie_exp.rs`	engine
Time stepping	`SYMPLECTIC_FLOW`: kick–drift–kick leapfrog whose drift is that exponential, `src/gauge/symplectic_flow.rs`	engine
Face angles of Table 2	`face_angles.py`, from the outputs of `examples/ricci_operator_probe.rs`	script on engine output
Arc (Manton) energy, tangent energy, the 2:1 blend	`verify.py`	script
Cayley map and Crank–Nicolson step	`verify.py`; on the mesh, `scripts/ricci_su2_pg_flow_cayley.py`	script
Haar sameness and Bertrand	`scripts/haar_sameness.py`, `verify.py`	script

The chord, as the engine computes it. The native $SU(2)$ action of Section 9 is the chord energy, exactly. Each face's holonomy is formed from its three links, and one is subtracted from its scalar part:

```
let (a, b, c) = [(0, 3, 1), (0, 4, 2), (1, 5, 2), (3, 5, 4)][face];
let mut h = product(product(u[a], u[b]), inverse(u[c]));
h[0] = h[0] - Jet::constant(1.);      // the chord: D = U_f - 1
...
energy = energy + Jet::constant(2. * mass[(i, j)]) * defects[i][k] * defects[j][k];
```

The variables are forward-mode jets, so the same pass returns all eighteen right-trivialised derivatives exactly. The face angles of Table 2 are read from what this code produced on the frozen meshes: the native probe `examples/ricci_operator_probe.rs`, built from the source above, whose executable is pinned in the receipts of that work by its SHA-256 (`71fb30db...`).

The engine steps along the arc, not the tangent. gigi's integrator, `SYMPLECTIC_FLOW`, moves each link by the exponential map, the arc, with a kick–drift–kick leapfrog driven by the Wilson force. The research flow of the curvature-recovery work [24] uses a Runge–Kutta–Munthe-Kaas method of order four, also built on the exponential, and at its finest level the Cayley step of Section 6 (`scripts/ricci_su2_pg_flow_cayley.py`). There the step was checked before use: its gap from the exponential falls as $|v|^3/12$, its time order is 1.99, and eight steps reproduce the

fourth-order runs to within 1.5% for a fifth to a third of the force evaluations. That is a research script. In the engine itself the Crank–Nicolson step is still a proposal, not a description.

What is not in the engine. The arc (Manton) energy, the tangent energy, the combination of Proposition 2 are computed only in `verify.py`. None is an engine verb, and none has been tested inside a simulation. The Cayley step has, in the research flow above, but it is not an engine verb either.

> In plain words. Some of this paper is already inside the engine we use: it reads how far each face has turned, measures the shortcut, and builds its energy from the shortcut, exactly as Section 9 assumes. Its clock moves things along the true curve. The detour and the two-to-one blend live only in the checking script for now. Putting the detour's step into the engine's clock is one of the open items below.

11 What a turn costs: toward Yang–Mills (speculative)

This section separates what is exact from what is a picture. The first two paragraphs are exact. The rest is a motivation: it rests on an assumption that the geometry does not supply, and it proves nothing about Yang–Mills theory.

The group of turns is curved (exact). The unit sphere in one dimension is the two points -1 and $+1$; in four dimensions it is the three-sphere of unit quaternions, which is the gauge group $SU(2)$ itself. Its curvature is how much its turns fail to commute: with the bi-invariant metric, the sectional curvature of the plane of X and Y is $\frac{1}{4}|[X, Y]|^2$ [17], which is 1 on the unit quaternions. For $SU(3)$ the same formula gives positive curvature on non-commuting planes and zero curvature on the planes spanned by two commuting directions, which $SU(3)$ has: it is curved, but it is not a round sphere. The photon's circle of turns, $U(1)$, commutes, and it is flat. The centre of $SU(2)$, the turns that commute with every other, is again exactly $\{-1, +1\}$, and -1 is the half turn of Section 3, where the outside path breaks.

The angle creates the energy (exact). A loop of the field that has not turned carries no energy. One that has turned through θ carries the energy of its chord, $1 - \cos\theta$ in Wilson's units (Section 4). This energy is continuous in θ: $1 - \cos\theta \sim \theta^2/2$ takes arbitrarily small positive values. The curvature of the group does not, by itself, give a least nonzero energy.

A motivating picture (speculative). In the ticket game of Section 8, a rule of the game makes the ticket indivisible: there is nothing to spend between no turn and one ticket. That indivisibility is an *added assumption*. It is not a consequence of the geometry above, which, as just noted, allows arbitrarily small energies. If some mechanism did make the smallest exchange indivisible, the result would have the shape of a floor under the energy, which is the shape a mass gap has; and the picture suggests that such a floor would be looked for in the gauge group rather than in the curvature of space, which flattens away as space grows. (For the same reason the eigenvalue 4 measured in the companion paper is not a mass gap: it comes from the curvature of the Poincaré homology sphere, not from the field.)

Status. The first two paragraphs are exact and checked. The picture is not: it assumes indivisibility, and it does not construct a quantum field theory. The Yang–Mills mass gap in flat, infinite space is the open Clay problem [20], which concerns the existence of a quantum Yang–Mills theory and a positive gap in its spectrum; this paper does not claim it, and the ticket game does not supply it. Related ideas, that curvature in the space of fields keeps its energy above zero, go back to Feynman [18] and Singer [19]. A companion note in the gigi repository (`docs/research/unit-curvature-2026-11-18.md`) works through the exact parts in more detail, with the related literature.

> In plain words. For the weak and strong nuclear forces, the turns themselves form a curved space ($SU(2)$ is a round sphere; $SU(3)$ is curved but has flat directions), and its curvature is how much the turns refuse to commute; for light, the turns commute and the space is flat. A turn is what costs energy, and the lattice pays for it along the straight shortcut, but a small turn costs only a little: the geometry alone puts no floor under the energy. The ticket game does have a floor, because its rules make a ticket whole. Whether nature has a rule like that for these forces is one of the great open problems (the mass gap), and this paper does not claim to have solved it.

12 What is claimed, and what is open

Claimed, each checked by `verify.py` (all checks passing; count in `verify_output.txt`):

- the length ordering, the 1/24 and 1/12 misses, and the Snellius–Huygens residual $\theta^5/320$;
- the chord's half-angle direction, and $S + d^2 = 1$ with d as half the chord;
- the Cayley construction in $\mathbb{C}$ and in the quaternions;
- the energy series and the cancellation of θ^4 in $(2C^2 + T^2)/3$;
- the three one-step rules' placement relative to the circle, and the Crank–Nicolson phase lag $s^3/12$;
- the lower twin's miss $\theta^5/2880$ and the 9:1 blend, the far-side sign flip, the continuation through the rim with $S + d^2 = 1$ intact, and Bertrand's three answers;
- the ticket game: the one-ticket angles (1 radian, 60°, 90°), the energy break-even at the right angle, the two whole-ticket corners O and $A + B$, the twin corner on the circle at 120°, and the tangent and twin corners meeting at 90°;
- the six-square grid of one-ticket angles, the Thales twin $2d^2 + 2S = 2$, the agreement of the three energy rules for small turns, and the Haar answer $\frac{1}{3} - \sqrt{3}/(4\pi)$ to Bertrand;
- the three time-stepping spirals: explicit and implicit Euler on mirror logarithmic spirals of pitch arctan $b(s)$, Crank–Nicolson on the circle, and the unique golden step $s^* \approx 0.6515$.

The face angles of Table 2 are measured from frozen data.

Not claimed. No priority is claimed for any classical identity above. We do not claim that the combined energy is a better lattice action; that would need simulations and the treatment of the half-turn barrier. We do not claim that this explains any error in the curvature-recovery work; Section 9 shows it does not.

Open.

- How the chord–tangent blend of Proposition 2 compares, in non-Abelian simulations, with the adjoint-improved action of [11] and with Lüscher-type barrier actions [13, 12], and whether its barrier at $U = -1$ inherits their known costs [9]. (The prior-art search is summarised in Section 1.)
- A Cayley (Crank–Nicolson) step inside the engine. It has been tested in the research flow of the companion paper [24]; gigi's own integrator steps with the exponential (Section 10).
- A strong-field or coarse-lattice test where θ is not small, the only regime where the energy combination could matter.

Reproducibility. `verify.py` and `face_angles.py` in this directory; the output of the first is saved as `verify_output.txt`, of the second as `face_angles.json`.

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