The Floor Theorem for the Yang–Mills Hessian at the Berger Field

A Floor Law Across the Berger Spheres

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Abstract. Let A be the Levi-Civita connection of a Berger metric $g = (t, 1, 1)$ on $S^3 = SU(2)$, viewed as an $SU(2)$ connection; it is not a Yang–Mills critical point. We study the Hessian of the Yang–Mills energy at A, compressed to the gauge-orthogonal slice, sector by sector in the Peter–Weyl decomposition. In the sector of spin $j/2$ an explicit field, the twist $f\,(\hat{\theta}^1 + i\hat{\theta}^2)\otimes(\tau_1 + i\tau_2)$, is an exact gauge-orthogonal eigenvector with eigenvalue $\lambda_t(j) = [j(j+2) - 2(t-3)j - 2(t-2)(t+4)]/t$. We prove that $\lambda_t(j)$ is the lowest eigenvalue of every sector for every $t \in [2.75, 10]$. A conserved charge splits each sector into blocks of size at most nine; interval inertia certificates cover the blocks of the sectors $j < 100$, and a closed-form Gershgorin bound covers $j \geq 100$. At $t = 3$, on the Poincaré homology sphere, this gives the gauge-orthogonal spectrum below 142 exactly; the first sector above the left-invariant one begins $168/3$ above the lowest mode, one third of the first nonzero scalar eigenvalue of the round unit Poincaré sphere. Across all circle-symmetric fields the twist remains an exact eigenvector, with eigenvalue $(x + (j+2)/\sqrt{t})^2 + 2\sqrt{t}\,x - y^2$, and it is the floor of every sector on the box $x \in [-3, -1.5]$, $y \in [0.5, 3]$, $t \in [3, 10]$. The construction reproduces the published round-sphere spectrum and Laquer's index. To our knowledge, these are the first exact eigenvalues of a Yang–Mills Hessian at a non-flat connection on a squashed three-sphere.

Keywords. Yang–Mills Hessian; Berger sphere; Poincaré homology sphere; spectral gap; Peter–Weyl decomposition; Weitzenböck identity; computer-assisted proof.
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1 Introduction

The second variation of the Yang–Mills energy decides whether a connection is a local minimum, how fast nearby fields grow or decay under the gradient flow, and what spectral observables a lattice discretization must reproduce. At Yang–Mills fields on round spheres and symmetric spaces it is classical. Bourguignon and Lawson [2] studied stability and isolation over the round sphere and other locally homogeneous spaces. Laquer [12] computed the index and nullity of the canonical connection over every simply connected compact irreducible symmetric space; for a group manifold the index is one (see also [16]). Around homogeneous backgrounds on the round S^3, Kumar, Lechtenfeld and Picanço Costa [10] diagonalized the fluctuation operator sector by sector in closed form. On Berger spheres, closed-form spectra are known for the scalar Laplacian [13], the Dirac operator [7, 1, 9] and the Hodge Laplacian on 1-forms [4].

We found no exact result for the Yang–Mills Hessian at a non-flat connection on a squashed three-sphere, nor at a connection that is not a Yang–Mills field. At the flat connection the Hessian

on coexact $\mathfrak{su}(2)$-valued 1-forms is d^*d tensored with $\mathfrak{su}(2)$, so its spectrum on a Berger sphere follows from the Hodge spectrum of [4]. This paper supplies one, at the natural candidate: the Levi-Civita connection of a Berger metric, viewed as an $SU(2)$ connection (the *Berger field*). It arises as the instantaneous state of a homogeneous gauge field transported along the Ricci flow of the metric [3], and it is the background on which lattice Yang–Mills forces on the Poincaré homology sphere were tested there. Its exact spectrum gives those computations a target.

The main theorem. The central result is the following (Theorem 6.7 in Section 6).

Theorem A. For every $t \in [2.75, 10]$ and every $j \geq 0$, the twist's eigenvalue $\lambda_t(j) = [j(j+2) - 2(t-3)j - 2(t-2)(t+4)]/t$ is the lowest eigenvalue of $H + GG^*$ in the sector of spin $j/2$, with multiplicity exactly two, and the lowest gauge-orthogonal eigenvalue there.

How the proof goes. (i) *Exactness:* the twist is alone in its block of a conserved charge, so $\lambda_t(j)$ is an exact eigenvalue (Proposition 6.1, Remark 6.6). (ii) *Reduction:* the same charge splits every sector into blocks of size at most nine, whose rows are known in closed form (Lemma 6.5). (iii) *Finitely many sectors:* for $j \leq 99$ every block is enclosed in interval arithmetic over slices of t and certified to lie above $\lambda_t(j)$ by exact counts of negative pivots. (iv) *All remaining sectors:* for $j \geq 100$ a closed-form Gershgorin quantity is proved positive by interval bisection. Each step has a script and a receipt (Section 8 and the code availability statement). The results are labelled throughout as exact identities, interval-certified inequalities, or numerical observations.

Main results. Section 2 fixes the operator: the Hessian H of the energy at the Berger field, against the L^2 mass, compressed to the gauge-orthogonal slice. The field is not at rest, so gauge directions are not zero modes, and the compression PHP is the object studied.

- *The left-invariant sector* (Section 3) is computed exactly. Three of its nine directions are gauge directions. Its slice spectrum is $-14/3$ (twice), $16 - 2\sqrt{73}$, $22/3$ (twice) and $16 + 2\sqrt{73}$. All three growth directions come from the commutator.
- *The floor formula* (Proposition 4.1): in every sector, $(j(j+2) - 14)/3$ is an exact eigenvalue, on an explicit eigenvector orthogonal to the gauge directions. That it is the bottom of its sector is proved for $j = 0, 12, 20, 24$ (Theorem 5.2), with a gap of $1/3$, and for every j by Theorem 6.7.
- *A Weitzenböck identity* (Lemma 5.1), proved in $U(\mathfrak{su}(2))$, gives a lower bound for every sector at once.
- *The Floor Theorem* (Theorem 5.2) follows, with certified inertia counts in the four sectors that matter.
- *The floor law* (Section 6): the formula extends to every Berger sphere $(t, 1, 1)$ as an exact eigenvalue (Proposition 6.1). It is certified as the floor at $t = 4$ (Theorem 6.4) and for every $t \in [2.75, 10]$ in every sector (Theorem 6.7); its eigenvector is a lowest Landau level, annihilated by the transverse raising operator, with its internal frame the lowest eigenvector of the curvature term of the Weitzenböck identity for $t \geq 12/7$, so on the whole certified range (Proposition 6.2), with the stretches at which each twist (the circular field of Proposition 4.1; terms in Section 2) goes free in closed form (Corollary 6.9).
- *Where the twist is the floor* (Section 7): across every circle-symmetric field on $(t, 1, 1)$ the twist's eigenvalue is in closed form (Proposition 7.1); it is the floor of every sector on a

box of fields (Theorem 7.3); for $H + GG^*$ it is not the floor of every sector at any $x > 0$; and where the block $B_{4,0}$ is the competitor, the edge is a root of an explicit polynomial (Proposition 7.2), which contains the Berger threshold t_A.

Section 6 follows the floor across the Berger family, and Section 7 across every circle-symmetric field. Section 8 checks the construction against the published round-sphere results and against lattice computations. Section 9 states the scope.

2 Setting

Frame and metric. On $S^3 = SU(2)$ let e_0, e_1, e_2 be a left-invariant frame with $[e_a, e_b] = 2\varepsilon_{abc}e_c$. The Berger metric is diagonal in it, with $g_0 = 3$ and $g_1 = g_2 = 1$; $\hat{e}_i = e_i/\sqrt{g_i}$ is orthonormal, with coframe $\hat{\theta}^i$ and structure constants $[\hat{e}_i, \hat{e}_j] = \sum_k c_{ij}^k \hat{e}_k$, $c_{ij}^k = 2\varepsilon_{ijk}\sqrt{g_k}/\sqrt{g_i g_j}$.

The Berger field. The Levi-Civita connection has Christoffel symbols $\Gamma_{ijk} = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j)$ in the orthonormal frame. Identifying $\mathfrak{so}(3)$ with $\mathfrak{su}(2)$, write it as the $\mathfrak{su}(2)$-valued 1-form $A = \sum_i M_i \hat{\theta}^i$ with constant M_i, where τ_0, τ_1, τ_2 is a basis of $\mathfrak{su}(2)$ with $[\tau_a, \tau_b] = \sum_c \varepsilon_{abc}\tau_c$ (the cross product, as for $\mathfrak{so}(3)$ under the identification; the frame brackets carry the factor 2, the Lie brackets do not), so that ad M_i acts on Lie components as $m_i \times \cdot$, τ_0 along the long axis. Its curvature F has components $F_{ij} = [M_i, M_j] - \sum_k c_{ij}^k M_k$. At the round metric A is a Yang–Mills field; at $g = (3, 1, 1)$ it is not.

Terms. A field is *at rest* if it is a Yang–Mills field, a critical point of the energy. A *growth mode* is a direction of negative second variation, a negative eigenvalue of the Hessian below. The *stretch* is the parameter t of the Berger metric $(t, 1, 1)$ of Section 6; the main case is $t = 3$. The *twist* of the sector of spin $j/2$ is the field $b = f\,(\hat{\theta}^1 + i\hat{\theta}^2) \otimes (\tau_1 + i\tau_2)$ of Proposition 4.1, with f the top weight along the long axis: it turns in the two short directions and, in step, in two internal directions. A twist *goes free* at the stretch where its eigenvalue crosses zero; beyond it, it is a growth mode. The *floor* of a sector is its lowest gauge-orthogonal eigenvalue, the bottom of the spectrum of PHP restricted to the sector (or, where stated, the lowest eigenvalue of $H + GG^*$ there), and a state *is the floor* if it attains it. In Section 7 the *edge* is the set of fields at which the twist's eigenvalue meets the lowest eigenvalue of another charge block, a level-crossing locus, and that block is the *competitor*.

The Hessian and the slice. A variation is $b = \sum_i b_i \hat{\theta}^i$. The second variation of $E = \frac{1}{2}\|F\|^2$ is the quadratic form

$$Q(b) = \sum_{i<j} \|D_{ij}b\|^2 + 2\langle F_{ij}, [b_i, b_j]\rangle, \qquad D_{ij}b = \hat{e}_i b_j - \hat{e}_j b_i + [M_i, b_j] - [M_j, b_i] - \textstyle\sum_k c_{ij}^k b_k, \quad (1)$$

and H is its operator against the L^2 mass $\sum_i \|b_i\|^2$. The gauge map is $G\xi = (\hat{e}_i\xi + [M_i, \xi])_i$ with L^2 adjoint G^*. Because A is not at rest, H does not preserve $\ker G^*$. The *gauge-orthogonal spectrum* is the spectrum of PHP on $\ker G^*$, with P the L^2 projection.

Sectors. Everything is invariant under left translations, so by Peter–Weyl the operators split into sectors indexed by $j \geq 0$. In the sector of spin $j/2$, $b_i \in V_j \otimes \mathfrak{su}(2)$, where V_j is the spin-$j/2$ representation (dimension $j + 1$) on which e_i acts by an anti-Hermitian ρ_i with $[\rho_a, \rho_b] = 2\varepsilon_{abc}\rho_c$. So H is a finite Hermitian matrix of size $9(j + 1)$. Write $J_0 = \frac{i}{2}\rho_0$, with weights $-\frac{j}{2}, \ldots, \frac{j}{2}$.

Conventions. With $[e_a, e_b] = 2\varepsilon_{abc}e_c$ the metric $(1,1,1)$ is the unit round sphere: its Ricci tensor is $2g$, and the scalar Laplacian has eigenvalue $j(j+2)$ on the sector of spin $j/2$. On $(t,1,1)$ the Ricci tensor is $\mathrm{diag}(2t, 4-2t, 4-2t)$ in the orthonormal frame. Eigenvalues are those of H defined by $\langle b, Hb\rangle = Q(b)$, with Q exactly as written in (1) and the L^2 mass $\sum_i \|b_i\|^2$; a convention that puts a factor $\frac{1}{2}$ in front of Q halves every eigenvalue, and scaling the metric by c multiplies every eigenvalue by $1/c$. With these normalisations the round-sphere spectrum is the published one (Section 8).

The Poincaré sphere. The binary icosahedral group $2I$ acts by left translations, and the frame, the metric and A descend to $S^3/2I$. The sector of spin $j/2$ survives with multiplicity $\dim V_j^{2I}$, given by the Molien series $(1+t^{30})/((1-t^{12})(1-t^{20}))$ of the binary icosahedral group, whose invariants go back to Klein [8]: the surviving sectors are $j = 0, 12, 20, 24, 30, 32, \ldots$. The first nonzero eigenvalue of the scalar Laplacian on the round unit Poincaré sphere is $j(j+2) = 168$ at $j = 12$ [11].

3 The left-invariant sector

At $j = 0$ the Hessian is a 9×9 matrix with entries in $\mathbb{Q}(\sqrt{3})$, computed in exact arithmetic.

Proposition 3.1. *At $j = 0$:*

1. *the spectrum of H is $-14/3$ (×2), $(29 - \sqrt{1345})/3$ (×2), $16 - 2\sqrt{73}$, 8, $(29 + \sqrt{1345})/3$ (×2), $16 + 2\sqrt{73}$;*
2. *the gauge directions are $b_i = [M_i, \xi]$, $\xi \in \mathfrak{su}(2)$, a three-dimensional space. The compression of H to it has eigenvalues 8, 12, 12. The value 8 is an eigenvalue of H with a purely gauge eigenvector; the two 12s are values of the compression only, since H does not preserve this space;*
3. *the gauge-orthogonal spectrum is $-14/3$ (×2), $16 - 2\sqrt{73}$, $22/3$ (×2), $16 + 2\sqrt{73}$.*

Proof. Exact symbolic computation of (1) and of the compression to the orthogonal complement of the gauge span (`scripts/ricci_su2_left_invariant_sector.py`, `scripts/ricci_su2_sector_minimality.py`). □

The eigenvalue 8 is a pure gauge direction, and the pairs $(29 \mp \sqrt{1345})/3$ are mixtures, 40% and 60% gauge. So the left-invariant sector has exactly three *physical* growth directions, $-14/3$ (twice) and $16 - 2\sqrt{73}$, each orthogonal to the gauge directions. On S^3 other sectors add more: at $t = 3$ the sectors $j = 1$ and $j = 2$ have four each, among them their twists (Corollary 6.9). On the Poincaré sphere those sectors do not survive, and the three are all. All of them come from the field acting on itself: with the commutator dropped, the same sector is $4/3$ (×6) and 12 (×3), all positive.

4 The floor formula

Proposition 4.1 (The floor formula)**.** *In the sector of spin $j/2$, let $f \in V_j$ be the top weight, $J_0 f = \frac{j}{2} f$. The field*

$$b = f\,(\hat{\theta}^1 + i\,\hat{\theta}^2) \otimes (\tau_1 + i\,\tau_2)$$

is an eigenvector of H, orthogonal to every gauge direction, with eigenvalue $\lambda(j) = (j(j+2) - 14)/3$.

Proof. At the top weight, $\hat{e}_0 f = -ijf/\sqrt{3}$, $J_+ f = 0$, $\rho_2 f = i\rho_1 f$ and $\|\rho_1 f\|^2 = j$. So every term of (1) on fields supported at the top weight involves only f and $u = \rho_1 f/\sqrt{j}$, with coefficients polynomial in j. On the squashed plane (frame and Lie indices $1, 2$, in the order $(1,1), (1,2), (2,1), (2,2)$) the Hessian is

$$\left(\tfrac{j^2}{3} + j\right) I + \frac{1}{3}\begin{pmatrix} 14 & 2ij & -ij & 28 \\ -2ij & 5 & -19 & -ij \\ ij & -19 & 5 & 2ij \\ 28 & ij & -2ij & 14 \end{pmatrix},$$

with eigenvalues $\frac{1}{3}(j^2 + 2j - 14)$, $\frac{1}{3}(j^2 + 4j - 14)$ and $\frac{1}{3}\left(j^2 + 3j + 33 \pm 3\sqrt{j^2 + 9}\right)$. The first has eigenvector $(-1, -i, -i, 1)$, which is b up to a factor. The other three are eigenvalues of this compression only, not of H. Three facts complete the argument, each an identity in j:

- the coupling of b to the only other top-weight direction it could reach, the τ_0 component of b_0, vanishes;
- $D_{ij} b$ has no component along u, because the two circular factors cancel, so H maps b into the top weight;
- $\langle G\xi, b\rangle = 0$ for every ξ, at every weight.

All of these are verified symbolically in j (`scripts/ricci_su2_sector_formula.py`). □

At $j = 0$ the proposition gives the growth mode $-14/3$. The circular field is the left-invariant growth mode lifted to the top weight of each higher sector, and the lift costs exactly $j(j+2)/3$.

5 The Weitzenböck bound and the gap

On the slice $G^* b = 0$, so $\langle b, Hb\rangle = \langle b, (H + GG^*)b\rangle$: adding the gauge term changes nothing there. The operator $H + GG^*$ has a useful form.

Lemma 5.1 (Weitzenböck identity)**.** *In every sector,*

$$H + GG^* = \sum_i X_i^* X_i + R_0, \qquad X_i = \rho_i/\sqrt{g_i} + \Omega_i,$$

where X_i is the covariant derivative (Ω_i is the action of the connection on the Lie index together with $-\Gamma_i$ on the frame index) and R_0 is a constant real symmetric 9×9 matrix with eigenvalues -12 (×2), $7 \pm \sqrt{73}$, $2 \pm 2\sqrt{13}$ (×2) and 8. Moreover $\omega^2 := \sum_i \|\Omega_i\|_2^2 = 76/3$ (operator norms).

Proof. Treat the ρ_i as generators of $U(\mathfrak{su}(2))$ with $[\rho_a, \rho_b] = 2\varepsilon_{abc}\rho_c$ and $\rho_i^* = -\rho_i$. Both sides are 9×9 matrices over $U(\mathfrak{su}(2))$ of filtration degree at most two. Put every product in Poincaré–Birkhoff–Witt order. Every coefficient of degree one and two then agrees, and the constant terms differ by R_0, in exact arithmetic with $\sqrt{3}$ symbolic. By the PBW theorem the identity holds in $U(\mathfrak{su}(2))$, hence in every representation V_j (`scripts/ricci_su2_gap_certificates.py`). □

Theorem 5.2 (The Floor Theorem)**.** *At the Berger metric $g = (3, 1, 1)$:*

1. *every gauge-orthogonal eigenvalue in the sector of spin $j/2$ is at least $\left(\sqrt{j(j+2)/3} - \omega\right)^2 - 12$, $\omega = \sqrt{76/3}$, whenever $\sqrt{j(j+2)/3} \geq \omega$;*

2. *for $j = 0, 12, 20, 24$, $\lambda(j)$ is the lowest gauge-orthogonal eigenvalue of its sector, and every other eigenvalue of $H + GG^*$ is at least $\lambda(j) + 1/3 - 10^{-53}$;*

3. *on the Poincaré sphere, the gauge-orthogonal spectrum below* 142 *consists of the spin-*0 *values of Proposition 3.1(3) and the spin-*6 *values from* 154/3 *upward; spin* 10 *begins exactly at* 142*.*

Proof. Part (1). Since $g_0 = 3 \geq g_1 = g_2 = 1$ and $-\sum_i \rho_i^2 = j(j+2)$ on V_j, $\sum_i \|\rho_i v\|^2/g_i \geq j(j+2)\|v\|^2/3$. Each term satisfies $\|X_i v\| \geq \big|\|\rho_i v\|/\sqrt{g_i} - \|\Omega_i v\|\big|$. The reverse triangle inequality in $\mathbb{R}^3$ gives $\sum_i \|X_i v\|^2 \geq (\sqrt{j(j+2)/3} - \omega)^2\|v\|^2$. With Lemma 5.1 and $\langle b, Hb\rangle = \langle b, (H + GG^*)b\rangle$ on the slice, this is (1).

Part (2). Write $A = H + GG^*$. By Proposition 4.1, $\lambda(j)$ is an eigenvalue of A of multiplicity at least two: the eigenvector b at the top weight, and its image at the bottom weight under the real structure of the integer-spin representation, which commutes with A because every coefficient is real. The rest is a rigorous count of the eigenvalues of A below $\mu = \lambda(j) + 1/3$ (`scripts/ricci_su2_gap_certificates.py`; every quantity below is recorded in its receipt).

1. *Enclosure.* $A - \mu$ is assembled in interval arithmetic, in mpmath's interval context (outward-rounded endpoints, 200 bits [14]), with $\sqrt{3}$ and every entry enclosed. It is realified, $\mathbb{C}^n \to \mathbb{R}^{2n}$ by $z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$, so every eigenvalue of A appears twice. Let M be the real symmetric matrix of the stored (rounded) midpoints of its lower triangle. The exact realified matrix is symmetric and each interval encloses its entry, so the two differ in norm by at most ρ, computed from the radii. Each radius bounds the distance from the interval to its stored midpoint, and is computed, like ρ itself, with outward rounding.

2. *Factorization.* M is factored without pivoting in 200-bit arithmetic. The stored factors, L unit lower triangular and D diagonal, define an exact matrix $LDL^\top$. Its distance from M is bounded a posteriori by $\|M - LDL^\top\|_F$, evaluated in interval arithmetic with outward rounding, so no rounding-error constant of the factorization enters.

3. *Inertia.* $LDL^\top$ is congruent to D, so by Sylvester's law $LDL^\top$ has exactly as many negative eigenvalues as D has negative pivots, and none equal to zero. Every pivot is recorded.

4. *Perturbation.* Let $\varepsilon = \rho + \|M - LDL^\top\|_F$, an upper bound evaluated in interval arithmetic (`scripts/ricci_su2_outward.py`). If D has k negative pivots, Weyl's inequality gives at most k eigenvalues of the realified $A - \mu$ below $-\varepsilon$, that is, at most $k/2$ eigenvalues of A below $\mu - \varepsilon$.

In each of the four sectors D has exactly four negative pivots, no pivot is smaller in absolute value than 0.63, and $\varepsilon < 7 \times 10^{-56}$ (Table 1). So A has at most two eigenvalues below $\lambda(j) + 1/3 - \varepsilon$, and both copies of $\lambda(j)$ are among them. Hence $\lambda(j)$ is the lowest eigenvalue of A, with multiplicity exactly two, and every other eigenvalue is at least $\lambda(j) + 1/3 - \varepsilon$. The same procedure at $\mu = \lambda(j) - 1/3$ finds no negative pivot, a consistency check that the argument does not need. The script refuses to certify if a pivot is zero, if a count differs, or if $\varepsilon \geq 1/3$. The eigenvector of Proposition 4.1 lies in the slice, so $\lambda(j)$ is the minimum there.

Part (3). At $j = 30$ the bound of (1) is $(\sqrt{320} - \omega)^2 - 12 = 153.26 > 142$; exactly, $\sqrt{320} - \sqrt{76/3} > \sqrt{154}$, with margin 0.45 in that form. The bound increases with j beyond. Part (2) covers the surviving sectors $j = 0, 12, 20, 24$, and Proposition 3.1 gives the spin-0 slice exactly. □

Table 1. The inertia certificates of Theorem 5.2(2), at $\mu = \lambda(j) + 1/3$. N: size of the realified matrix; ρ: enclosure radius; residual: $\|M - LDL^\top\|_F$; $\varepsilon = \rho +$ residual. At $\mu = \lambda(j) - 1/3$ no sector has a negative pivot, $\varepsilon \le 6.2 \times 10^{-56}$ and the smallest pivot is at least 0.63.

j	$\lambda(j)$	N	ρ	residual	ε	neg. pivots	$\min \lvert d_k \rvert$
0	$-14/3$	18	3.0×10^{-58}	5.5×10^{-59}	3.5×10^{-58}	4	0.679
12	$154/3$	234	7.7×10^{-57}	6.3×10^{-57}	1.4×10^{-56}	4	21.3
20	142	378	2.4×10^{-56}	1.9×10^{-56}	4.3×10^{-56}	4	60.0
24	$610/3$	450	3.7×10^{-56}	2.9×10^{-56}	6.6×10^{-56}	4	66.9

Remark 5.3. The double-precision computation of $H + GG^*$ for every $j \le 120$ gives the same picture: bottom exactly $\lambda(j)$, twice, and the next eigenvalue at least $\lambda(j) + 2/3$. The theorem does not rely on this. Minimality for $j > 120$ is not needed for the gap; it holds for every j by Theorem 6.7 at $t = 3$.

In one line, with $\lambda_j(\Delta_{S^3}) = j(j+2)$ the eigenvalue of the scalar Laplacian of the round unit sphere on the sector of spin $j/2$, the floor formula reads

$$\boxed{\lambda_{\text{floor}}(j) \;=\; \frac{\lambda_j(\Delta_{S^3}) - 14}{3} \;=\; \frac{j(j+2) - 14}{3}}$$

It is an eigenvalue in every sector (Proposition 4.1); the Floor Theorem shows that it is the floor of every sector below $j = 30$ that survives on the Poincaré sphere, and bounds every higher sector; Theorem 6.7 shows that it is the floor of every sector. On the Poincaré sphere the floors of the first two surviving sectors differ by $\lambda_1(\Delta_{S^3/2I})/3 = 168/3$.

Corollary 5.4 (The floor gap)**.** *On the Poincaré sphere, the first sector above the left-invariant one begins* $\lambda(12) - \lambda(0) = 168/3 = 56$ *above the growth mode: one third of the round sphere's first scalar eigenvalue. The spectral gap between the two sectors runs from* $16 + 2\sqrt{73} = 33.09$ *to* $154/3 = 51.33$*. The first number is a difference between sector floors and the second an empty interval of width* 18.25*; neither is a quantum mass gap.*

6 The floor across the Berger family

The computation above never used the value 3 except through g_0. Replace it by any stretch $t > 0$: the Berger metric $g = (t, 1, 1)$, its Levi-Civita field, and the same circular top-weight field b.

Proposition 6.1 (The floor law)**.** *For every* $t > 0$ *and every* $j \ge 0$*, the field* $b = f\,(\hat\theta^1 + i\hat\theta^2) \otimes (\tau_1 + i\tau_2)$ *is an eigenvector of* H *at the Berger field of* $(t, 1, 1)$*, orthogonal to every gauge direction, with eigenvalue*

$$\boxed{\lambda_t(j) \;=\; \frac{j(j+2) - 2(t-3)\,j - 2(t-2)(t+4)}{t}}$$

At $t = 3$ *this is* $\lambda(j)$ *of Proposition 4.1.*

Proof. The proof of Proposition 4.1, with t symbolic in place of 3. The block on the squashed plane has eigenvalues $\lambda_t(j)$, $(j^2+4jt-8j-2t^2-4t+16)/t$ and $(j^2+jt+5t^2-4t \pm t\sqrt{j^2+t^2})/t$ (the last three of

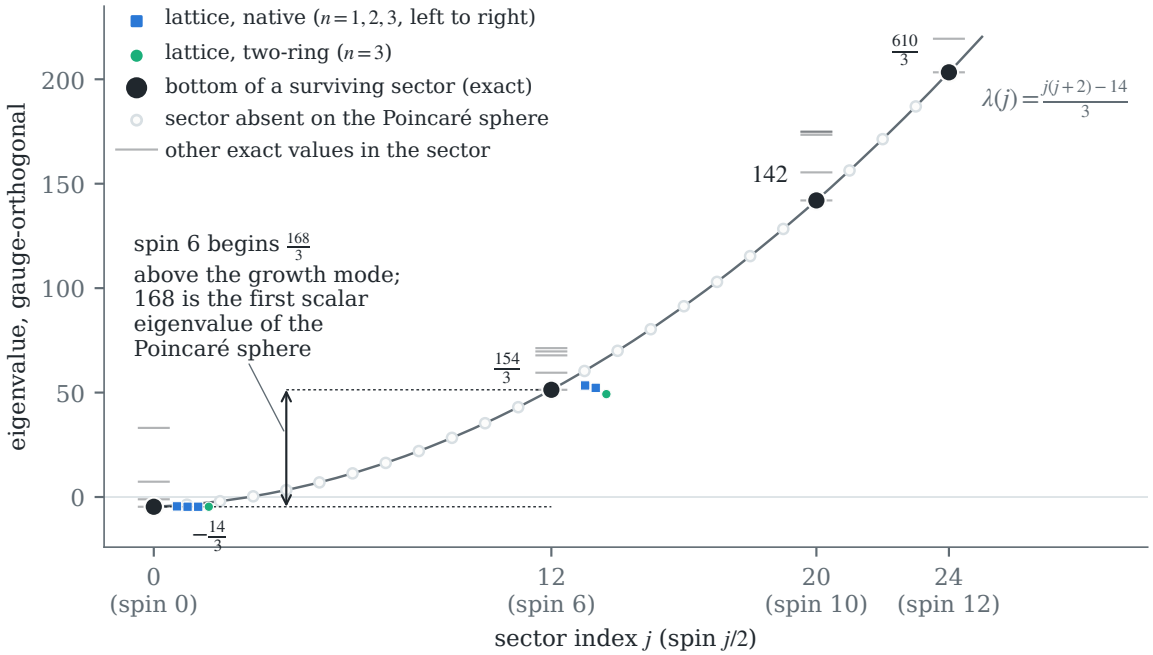

FIGURE 1. The gauge-orthogonal spectrum at the Berger field, sector by sector. Curve: $\lambda(j) = (j(j+2) - 14)/3$. Filled: the bottom of each sector that survives on the Poincaré sphere; hollow: sectors the icosahedral group forbids; grey ticks: the other exact values of a surviving sector. Squares and circles: lattice eigenvalues near the left-invariant values and the lowest gauge-orthogonal values of spin 6 from the two lattice forces of [3] (native $n = 1, 2, 3$; two-ring $n = 3$).

the compression only). The first has eigenvector $(-1, -i, -i, 1)$, and the three facts listed there (no coupling to the τ_0 component of b_0, no leak into the next weight, and orthogonality to every gauge direction) hold as identities in j and t (`scripts/ricci_su2_sector_formula_squashed.py`). □

The Weitzenböck identity of Lemma 5.1 holds on every Berger sphere, and it explains the shape of the floor law.

Proposition 6.2 (The twist is a lowest Landau level)**.** *Let $t > 0$.*

1. *In every sector, $H + GG^* = \sum_i X_i^* X_i + R_0(t)$ with $X_i = \rho_i/\sqrt{g_i} + \Omega_i$, where $R_0(t)$ is a constant real symmetric 9×9 matrix with eigenvalues $12 - 8t$ (×2), $4(t-1)$, $-2\big(\sqrt{2t^2 - 2t + 1} - 1\big)$ (×2), $2\big(\sqrt{2t^2 - 2t + 1} + 1\big)$ (×2) and $3t - 2 \pm \sqrt{9t^2 - 4t + 4}$. For $t \geq 12/7$, $12 - 8t$ is the lowest of them.*
2. *Write $b = f \otimes \beta$, with β the internal frame $(\hat\theta^1 + i\hat\theta^2) \otimes (\tau_1 + i\tau_2)$, and $X_\pm = X_1 \pm iX_2$. Then*

$$X_+ b = 0, \qquad X_0 b = i\kappa\, b, \quad \kappa = \frac{2t - 4 - j}{\sqrt{t}}, \qquad R_0(t)\beta = (12 - 8t)\beta.$$

3. *Consequently, for $\|b\| = 1$, in the identity of (1), the fibre term of b is $\|X_0 b\|^2 = \kappa^2$, the curvature term is $12 - 8t$, and the transverse term is $\|X_1 b\|^2 + \|X_2 b\|^2 = 2(j + t)$:*

$$\lambda_t(j) = \kappa^2 + 2(j + t) + (12 - 8t).$$

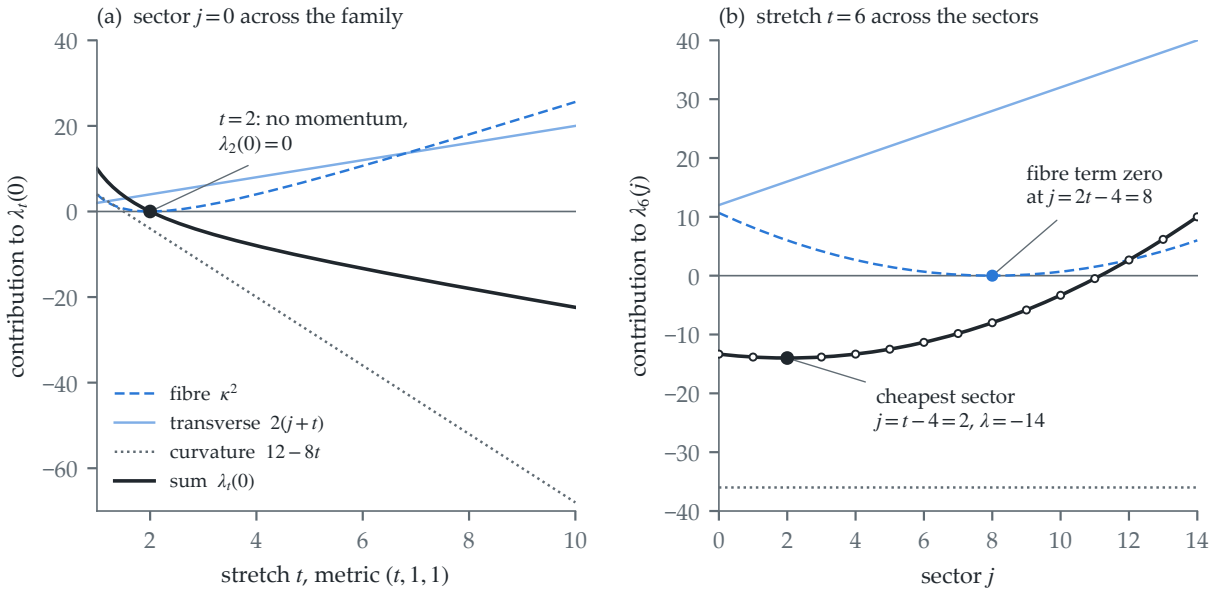

Figure 2. The three pieces of Proposition 6.2(3): fibre κ^2 (dashed), transverse $2(j+t)$, curvature $12-8t$ (dotted), and their sum $\lambda_t(j)$. (a) The sector $j=0$ across the family: the momentum κ vanishes at $t=2$, where $\lambda_2(0)=0$. (b) The stretch $t=6$ across the sectors (open circles): the fibre term is smallest at $j=2t-4=8$, and the transverse term moves the minimum of $\lambda_6(j)$ to $j=t-4=2$, with value $4-3t=-14$ (Corollary 6.9). Exact; the identity is checked symbolically in `make_figure_three_pieces.py`.

Proof. (1) The proof of Lemma 5.1 with t symbolic: every coefficient of degree one and two agrees in $U(\mathfrak{su}(2))$, identically in t (`scripts/ricci_su2_weitzenbock_symbolic_t.py`). The eigenvalues are those of the symbolic $R_0(t)$, and the comparison is elementary. (2) $(\rho_1+i\rho_2)f=0$ because f is the top weight, and $(\Omega_1+i\Omega_2)\beta=0$, $\Omega_0\beta=i(2t-4)\beta/\sqrt{t}$ and $R_0(t)\beta=(12-8t)\beta$ hold as identities in t; $\rho_0 f=-ijf$ (`scripts/ricci_su2_landau_symbolic.py`). (3) follows from (1), (2) and Proposition 6.1. □

Remark 6.3. In the language of a charged particle in a magnetic field, $X_+b=0$ is the lowest-Landau-level condition for the two transverse covariant derivatives, κ is the momentum along the fibre, and $12-8t$ is the most negative curvature the internal frame can see (for $t\geq 12/7$). The fibre term alone is smallest where the momentum vanishes, at $j=2t-4$; the transverse term $2j$ moves the minimum of $\lambda_t(j)$ over j to $j=t-4$ (Corollary 6.9). The statement that b is the *floor* is not implied by (2): it needs every other state of the sector to cost more, which is Theorem 6.7. The same three pieces, a momentum along the fibre, a lowest Landau level across it and a spin term from $-2[F,\cdot\,]$, make up the unstable mode of Nielsen and Olesen in a constant chromomagnetic field on flat space [15]. We note the resemblance only; no limit relating the two is proved here.

The certificate of Theorem 5.2 takes t as a parameter; run at $t=3$ it reproduces Table 1 exactly (`scripts/ricci_su2_gap_certificates_t.py`). At a second squashing it gives the same kind of complete statement.

Theorem 6.4 (The Floor Theorem at $(4,1,1)$)**.** *At the Berger metric $g=(4,1,1)$:*

1. *the Weitzenböck identity of Lemma 5.1 holds with R_0 of eigenvalues* -20 (×2), -8 (×2), 12 (×3)

and $10 \pm 2\sqrt{33}$*, and* $\omega^2 = 36$*; every gauge-orthogonal eigenvalue in the sector of spin* $j/2$ *is at least* $(\sqrt{j(j+2)/4} - 6)^2 - 20$ *whenever* $\sqrt{j(j+2)/4} \geq 6$*;*

2. *for* $j = 0, 12, 20, 24, 30, 32$, $\lambda_4(j) = j^2/4 - 8$ *is the lowest gauge-orthogonal eigenvalue of its sector, and every other eigenvalue of* $H + GG^*$ *is at least* $\lambda_4(j) + 1/3 - 10^{-53}$*;*

3. *on the Poincaré sphere, the gauge-orthogonal spectrum below* 136 *consists of the spin-*0 *values* -8 $(\times 2)$, $22 - 2\sqrt{129}$, $29/5$ $(\times 2)$ *and* $22 + 2\sqrt{129}$*, the spin-*6 *values from* 28 *upward and the spin-*10 *values from* 92 *upward; spin* 12 *begins exactly at* 136*.*

Proof. As for Theorem 5.2. The six sectors certify with exactly four negative pivots each, $\varepsilon < 1.3 \times 10^{-55}$ and smallest pivot at least 0.649. The tail bound exceeds 136 from $j = 36$ on: exactly, $\sqrt{342} - 6 > \sqrt{156}$, with margin 0.0032 in that form. The spin-0 slice is computed in exact arithmetic (receipts in `research/su2-floor-family-20261119/`). □

So at $t = 4$ the Poincaré sphere has still exactly three growth directions, and the floor gap is 36. Unlike $t = 3$, spin 6 begins below the top of the spin-0 spectrum, so the sectors interleave there.

The floor law over a range. A certificate need not be taken one t at a time. For a slice $t \in [a, b]$, enclose $A(t) - \lambda_t(j) - \delta$ for *all* t in the slice at once, with t and $\lambda_t(j)$ entered as intervals, and proceed as in the proof of Theorem 5.2. Bound the perturbation by $\rho = \min(\text{Frobenius, maximal row sum})$ of the enclosure radii; both bound the 2-norm of a symmetric perturbation. As in Theorem 5.2, the radii are taken about the stored midpoints, and ρ, the residual $\|M - LDL^\top\|_F$ and ε are evaluated with outward rounding. Then exactly four negative pivots and $\varepsilon < \delta$ give, for every t in the slice, at most two eigenvalues below $\lambda_t(j) + \delta - \varepsilon$. By Proposition 6.1, both copies of $\lambda_t(j)$ are among them.

Charge blocks. The rotation about the long axis, acting at once on the sector, the frame index and the Lie index, is a symmetry of the Berger field. Its charge makes the certificate independent of the size of the sector. Take $\rho_0, \rho_1, \rho_2 = -2iJ_z, -2iJ_x, -2iJ_y$ on V_j (equivalent to the representation used so far: same brackets), with weights $m \in \{-l, \dots, l\}$, $l = j/2$, and take the frame and Lie indices in the eigenbasis of the rotation, with charges $c_f, c_l \in \{-1, 0, 1\}$.

Lemma 6.5 (Charge blocks)**.** *Let* $t > 0$ *and* $c = c_f + c_l$*. In this basis* $\Omega_0 = i\,\mathrm{diag}\big(c(t-2)/\sqrt{t}\big)$*,* Ω_+ *raises* c *by exactly one,* Ω_- *lowers it by exactly one, every nonzero entry of* $\Omega_\pm$ *has modulus* $\sqrt{2t}$*, and* $R_0(t)$ *preserves* c*. Hence* $H + GG^*$ *preserves the charge* $q = m + c$*, and the sector of spin* $j/2$ *is the orthogonal sum of the charge blocks, each spanned by the at most nine states* (m, c_f, c_l) *with* $m = q - c_f - c_l$*,* $|m| \leq l$*. The state of charge* $l + 2$ *is the twist* b *alone, and the state of charge* $-l - 2$ *its conjugate.*

Proof. Exact, with t symbolic (`scripts/ricci_su2_charge_selection_symbolic.py`). For every j, the antiunitary map $J \otimes K$, with J the real (j even) or quaternionic (j odd) structure of V_j and K complex conjugation on the frame and Lie indices, commutes with $H + GG^*$, whose coefficients are real, and reverses the charge; so the block of charge $-l - 2$ has the same entry as the twist's. Since ρ_0 is diagonal and $\rho_1 \pm i\rho_2 = -2iJ_\pm$ shifts m by ± 1, each of X_0 and $X_\pm$ shifts q by 0 or ± 1, and $\sum_i X_i^* X_i = X_0^* X_0 + \frac{1}{2}(X_+^* X_+ + X_-^* X_-)$ preserves q; so does $R_0(t)$. □

Remark 6.6 (Why the floor formula is exact). Lemma 6.5 explains Propositions 4.1 and 6.1, which were proved by symbolic computation. The twist is the only state of charge $l+2$, so it spans a 1×1 block and is an eigenvector of $H+GG^*$. The gauge map preserves the charge, with the gauge parameter ξ carrying the weight and the Lie index only, so its charges reach $l+1$ at most. Hence $G^*b=0$, and $Hb=(H+GG^*)b$: the twist is an exact gauge-orthogonal eigenvector of H, and its eigenvalue is its diagonal entry. The argument uses only the rotation about the long axis, so it holds for any left-invariant background with that symmetry. Whether the twist is the *floor* is a separate statement, and it depends on the background (Section 6, "Further observations").

Theorem 6.7 (The floor law on $2.75\le t\le 10$)**.** *For every $t\in[2.75,10]$ and every $j\ge 0$, $\lambda_t(j)$ is the lowest eigenvalue of $H+GG^*$ in the sector of spin $j/2$, with multiplicity exactly two, and the lowest gauge-orthogonal eigenvalue there.*

Proof. By Lemma 6.5 it suffices to show that every charge block other than the two twist blocks has all its eigenvalues above $\lambda_t(j)$; the two twist blocks are 1×1 with entry $\lambda_t(j)$ (Proposition 6.1).

Sectors $j\le 99$. Each block is enclosed for all t in a slice at once and certified positive definite after the shift $\lambda_t(j)+\eta$, as described above ($\eta>0$ plays the role of δ there, chosen per block and slice as half the double-precision gap), with no negative pivot and $\varepsilon<\eta$ (`scripts/ricci_su2_block_interval_cert.py`). For $j\le 40$, 943 blocks and 12,596 block-slices, smallest margin $\eta-\varepsilon=1.7\times 10^{-5}$ (receipt `block_cert_j40.json`); for $41\le j\le 99$, 4,307 blocks and 36,166 block-slices, smallest margin 2.9×10^{-3} (receipt `block_cert_j41_99.json`).

Sectors $j\ge 100$. Write $m=l-x$. The row of (m,c_f,c_l) and the row of $(-m,-c_f,-c_l)$ have the same quantity G below ($x\mapsto 2l-x$, exactly), so $0\le x\le l$ suffices. For the state of type (c_f,c_l), the diagonal entry of its block minus $\lambda_t(j)$ minus the sum of the moduli of its off-diagonal entries (its Gershgorin radius) is, identically in l, x and t,

$$G=4x(2l-x)\Big(1-\frac{1}{t}\Big)+\frac{4l(2-c)(t-2)}{t}+E-2\sqrt{2t}\Big[n_-\sqrt{x(2l-x+1)}+n_+\sqrt{(2l-x)(x+1)}\Big],$$

with $E=c^2(t-2)^2/t+4c(t-2)x/t+C_a-R_a+2(t-2)(t+4)/t$, where C_a is the diagonal entry of $-(\Omega_1^2+\Omega_2^2)+R_0(t)$, R_a the sum of the moduli of its other entries in the same row, and $n_\pm$ the numbers of nonzero entries of $\Omega_\pm$ in the column of the type (`scripts/ricci_su2_charge_selection_symbolic.py`; the nine types are tabulated in `scripts/ricci_su2_floor_large_j.py`). If $G>0$ on every row except the twist, every other block lies above $\lambda_t(j)$ by Gershgorin's theorem. For $l\ge 50$ and $t\in[2.75,10]$ this is proved by bisection in interval arithmetic: for $x\le 20$ on $(1/l,t)$, $G/l>5.9\times 10^{-4}$ (373 boxes); for $x\ge 20$, after dividing by lx, on the relaxed box $(x/l,1/l,1/x,t)\in[0,1]\times[0,1/50]\times[0,1/20]\times[2.75,10]$, $G/(lx)>0.85$ (9 boxes) (receipt `floor_large_j.json`).

Receipts are in `research/su2-floor-family-20261119/`. Independently, a first certificate on the full sector matrices of the fifteen sectors $j\le 12$, 20, 24 agrees: in each of the fifteen sectors the slices tile $[2.75,10]$ exactly (1,091 slices in all), the finest 0.002 wide near $t=2.75$, where the gap above the floor is smallest (0.16 for $j=4$, 0.18 for $j=1$). Every slice has exactly four negative pivots and $\varepsilon<\delta$, the smallest margin $\delta-\varepsilon$ being 0.0018. A degenerate slice at $t=3$ reproduces Table 1 (`scripts/ricci_su2_floor_interval_cert.py`, receipt `research/su2-floor-family-20261119/interval_cert_2.75_10.json`). □

At $t = 3$ this completes Proposition 4.1: $(j(j+2) - 14)/3$ is the floor of every sector of the Berger field $(3, 1, 1)$, not only of those that survive on the Poincaré sphere.

Remark 6.8 (Below the range). There are two thresholds, one for each statement of Theorem 6.7. *Gauge-orthogonal spectrum.* In double precision, $\lambda_t(j)$ is its bottom in every sector $j \le 40$ for every $t \in [2.7, 2.75]$ tested. Below $t^* \approx 2.6980$ (located by bisection) a lower eigenvalue appears, first in the sector $j = 4$; at $t \le 2$ the branch is the floor of no sector. *The operator $H + GG^*$.* The threshold is algebraic. The charge-0 block of the sector $j = 4$ has $\lambda_t(4)$ as an eigenvalue exactly when $s = \sqrt{t}$ is a root of its determinant, whose largest positive root is that of

$$3s^{16} + 121s^{14} + 1066s^{12} - 2836s^{10} - 18624s^8 + 20992s^6 + 181248s^4 - 425984s^2 + 262144,$$

$t_A = s^2 = 2.7311222782\ldots$ (`scripts/ricci_su2_tstar_exact.py`). In double precision, below t_A that block has an eigenvalue under $\lambda_t(4)$, and between t_A and 2.75 no sector $j \le 40$ has one (`scripts/ricci_su2_tA_scan.py`). So between t^* and t_A the eigenvalue of $H + GG^*$ below $\lambda_t(4)$ is not gauge-orthogonal, and the twist remains the floor of the slice. Only the range of Theorem 6.7 is certified.

Corollary 6.9. *Where $\lambda_t(j)$ is the floor:*

1. The floor gap. *On the Poincaré sphere, $\lambda_t(12) - \lambda_t(0) = (240 - 24t)/t$: $168/3$ at $t = 3$, 36 at $t = 4$, and 0 at $t = 10$. The relation to the first scalar eigenvalue holds at $t = 3$ only, and characterises it (Remark 6.10).*

2. Where a twist goes free. $\lambda_t(j) = 0$ *exactly at*

$$t_j = \tfrac{1}{2}\Big(\sqrt{(j+2)^2 + 2(j+4)^2} - (j+2)\Big),$$

   and $\lambda_t(j) < 0$ for $t > t_j$. The values are $t_3 = 3.045$, $t_4 = 3.403$, inside the range of Theorem 6.7. The same formula gives $t_0 = 2$, $t_1 = 2.341$ and $t_2 = 2.690$. These are zeros of the branch $\lambda_t(j)$ outside the certified range: at t_0 the branch is not the floor (Remark 6.8), the double-precision scan already finds a lower eigenvalue in the sector $j = 1$ at $t = 2.5 > t_1$, and at t_2 that the branch is the floor is numerical only. At stretch t, every sector with $j < (t-4) + \sqrt{3t^2 - 4t}$ has negative floor. Each floor crosses zero linearly in $t - t_j$.

3. The softest sector moves. *Over j, $\lambda_t(j)$ is smallest near $j = t - 4$.*

Proof. Algebra on the formula of Proposition 6.1; the statements about floors hold where Theorem 6.7 or Theorem 6.4 applies. □

Remark 6.10 (The Casimir point). The cost of lifting the twist from the left-invariant sector to the sector of spin $j/2$ splits as

$$\lambda_t(j) - \lambda_t(0) = \Big(\frac{j^2}{t} + 2j\Big) - \frac{4(t-2)}{t}\, j.$$

The first term is the eigenvalue of the scalar Laplacian of $(t, 1, 1)$ on the top-weight vector f (since $-\rho_0^2 f = j^2 f$ and $-\sum_i \rho_i^2 = j(j+2)$); the second couples the weight along the long axis to the stretch. The lift equals the round Casimir divided by the stretch, $j(j+2)/t$, for every j if and only if $t = 3$. So at $t = 3$ the floor gap on the Poincaré sphere is $\lambda_1(\Delta_{S^3/2I})/3$ because there the lift depends on j only through the Casimir. At $t = 2$ the coupling vanishes: the lift is exactly the scalar Laplacian on f, and $\lambda_2(0) = 0$.

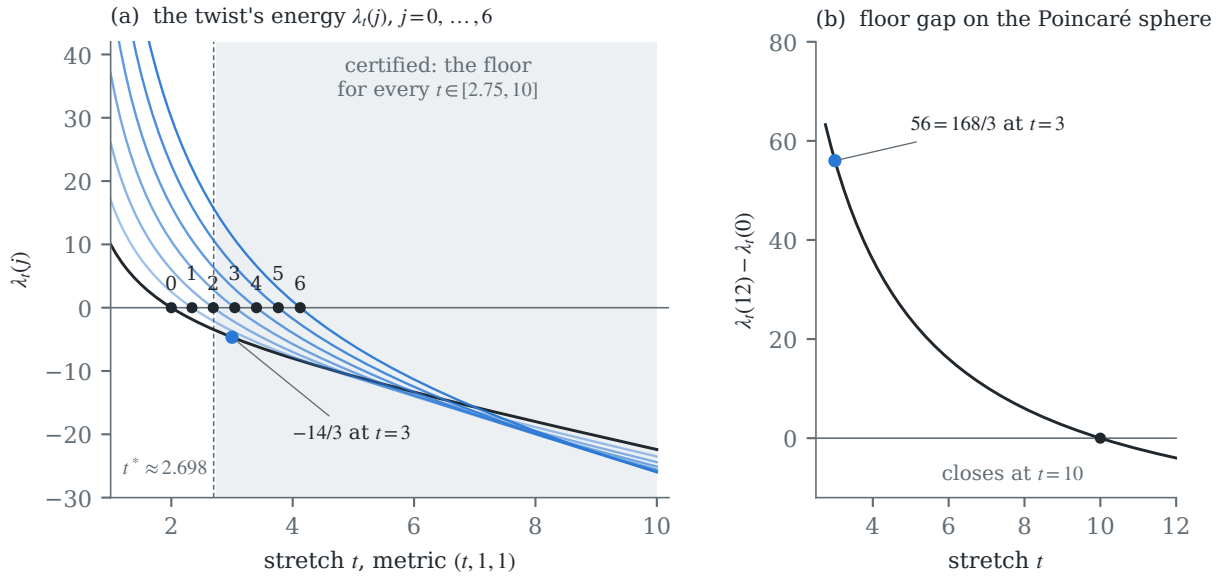

Figure 3. The floor law across the Berger family. (a) $\lambda_t(j)$ for $j = 0, \dots, 6$; dots mark where each twist goes free (t_j); shaded: the range of Theorem 6.7; dashed: t^* of Remark 6.8. (b) The floor gap on the Poincaré sphere, $(240 - 24t)/t$.

Further observations (double precision; not certified). Two numerical experiments locate what the floor law is and is not (scripts and receipts in the repository; see the availability section).

- *Landscape, not dynamics.* Under the gauge-fixed linearised flow with the operator held at the Berger field, a disturbance in sector j decays for $t < t_j$, grows for $t > t_j$, and turns into the twist (overlap 1.000 in every run). Under the actual Yang–Mills flow the Berger field is not at rest. It collapses towards a flat field within flow time 0.1 (energy 34.6 → 0.73 at $t = 3.55$), faster than any twist grows, and the disturbance decays. The floor law describes the second variation at the Berger field. It is not by itself a dynamical transition.
- *Resting fields.* Among left-invariant fields on $(t_1, t_2, 1)$, a multistart search (36 shapes, axis ratios up to 20) finds three clusters of Yang–Mills fields up to constant gauge rotation: the flat field $A = 0$; the flat pure-gauge field $A = g^{-1}dg$, which is gauge-equivalent to 0 under a non-constant gauge transformation but not under a constant one; and one non-flat relative of the meron. A multistart search is evidence, not a classification. At every stretch checked, the relative has exactly one negative eigenvalue of $H + GG^*$ among the sectors tested, and it lies in the left-invariant sector: for $j \le 12$ at $t = 0.05, 0.10, \dots, 10$ along $(t, 1, 1)$, and for $j \le 8$ on the 36 uneven shapes. No eigenvalue of the sectors $1 \le j \le 12$ is negative at these points; crossings between the sampled stretches are not excluded.
- *Symmetry does not make the floor.* The squashed meron has the same rotation symmetry, so by Remark 6.6 the twist is an exact gauge-orthogonal eigenvector there too. Yet on the grid $t = 0.05, 0.10, \dots, 10$ it is the floor of no sector $j \le 12$ at any point. For $t \le 1$ it is the top of every sector $j \le 12$, and it descends as t grows without reaching the bottom. On the Berger field the same scan finds it the floor of every sector $j \le 12$ at exactly the grid points $t \ge 2.70$. The difference lies in the curvature term. In the Weitzenböck form [2],

$R_0 = \mathrm{Ric} - 2[F, \cdot\,]$, with $([F, \cdot\,]b)_i = \sum_j [F_{ij}, b_j]$: the Ricci curvature plus the field strength with the factor 2 of the Hessian (checked to 4×10^{-15} on both fields). On the twist's internal frame the Ricci share is $4 - 2t$ on both fields. The field-strength share is $8 - 6t$ on the Berger field (total $12 - 8t$), but positive on the squashed meron, between 0.4 and 1.3 for t from 2 to 8 (`scripts/ricci_su2_test2_family.py`, `scripts/ricci_su2_gyromagnetic_split.py`, `scripts/ricci_su2_meron_ceiling_check.py`).

7 Where the twist is the floor

The Berger field is one member of a larger family. Up to a constant gauge transformation, every left-invariant field on $(t, 1, 1)$ that is symmetric under the joint rotation about the long axis has the form $M_0 = x\,\tau_0$, $M_1 = y\,\tau_1$, $M_2 = y\,\tau_2$ with $y \geq 0$ (a commutant computation, `scripts/edge_wp4_design_checks.py`): a strength x along the long circle and a strength y across it. The Berger field is $x = (2 - t)/\sqrt{t}$, $y = \sqrt{t}$; the multiples $A = s\theta$ of the Maurer–Cartan form are $x = 2s/\sqrt{t}$, $y = 2s$; the squashed meron lies in the family too. By Remark 6.6 the twist is an exact gauge-orthogonal eigenvector of every member.

Proposition 7.1 (The twist across the family)**.** *For every $t > 0$, $x, y \in \mathbb{R}$ and $j \geq 0$, the twist's eigenvalue is*

$$\lambda(x, y, t, j) = \left(x + \frac{j+2}{\sqrt{t}}\right)^2 + 2\sqrt{t}\,x - y^2.$$

On the Berger field it is $\lambda_t(j)$. The cost of lifting the twist from the sector $j = 0$ to the sector j is $(j^2 + 4j)/t + 2xj/\sqrt{t}$; the second term lowers it if and only if $x < 0$, that is, when the field along the long circle turns against the twist's own rotation.

Proof. The twist is alone in its charge block, so its eigenvalue is the diagonal entry of $H + GG^*$ there: by the Weitzenböck form, $\|X_0 b\|^2 + \|X_1 b\|^2 + \|X_2 b\|^2 + \langle \beta, R_0 \beta \rangle$ for $\|b\| = 1$, with $R_0 = \mathrm{Ric} \otimes I - 2[F, \cdot\,]$. Symbolic in x, y, t and checked against the general builder at 40 random points to 1.4×10^{-15} (`scripts/ricci_su2_twist_family_closed_form.py`). □

The rows of every charge block are likewise in closed form in (x, y, t), so whether the twist is the floor is a comparison between λ and the lowest eigenvalue of finitely many competing blocks.

Proposition 7.2 (The edge)**.** *Let $P(x, y, \sqrt{t})$ be the numerator of $\det\big(B_{4,0} - \lambda(x, y, t, 4)\big)$, with $B_{4,0}$ the charge-0 block of the sector $j = 4$. Over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ it factors as $P = P_1 P_2$, with P_2 of degree 7, 8 and 16 in x, y and $s = \sqrt{t}$. On the Berger path $x = (2 - s^2)/s$, $y = s$, the factor P_2 contains the polynomial of Remark 6.8 whose root is t_A.*

Proof. Exact symbolic determinant and factorization, and exact polynomial division on the Berger path (`scripts/ricci_su2_edge_polynomial.py`). □

The edge, numerically. In the sectors $j \leq 12$, at every (t, y) computed, the twist is the floor of $H + GG^*$ for x below an edge $x^*(t, y)$ and not above it, within the sampled window (the region is also bounded below; see the end of this paragraph), and the same holds for the gauge-orthogonal spectrum with a slightly higher edge. Where the block $B_{4,0}$ is the active competitor, the computed edge x^* is a root of P_2 (to 10^{-8}). Other parameter ranges are controlled by other charge blocks: at small y a block of the sector $j = 2$ or 3, and at 4 of 38 held-out points

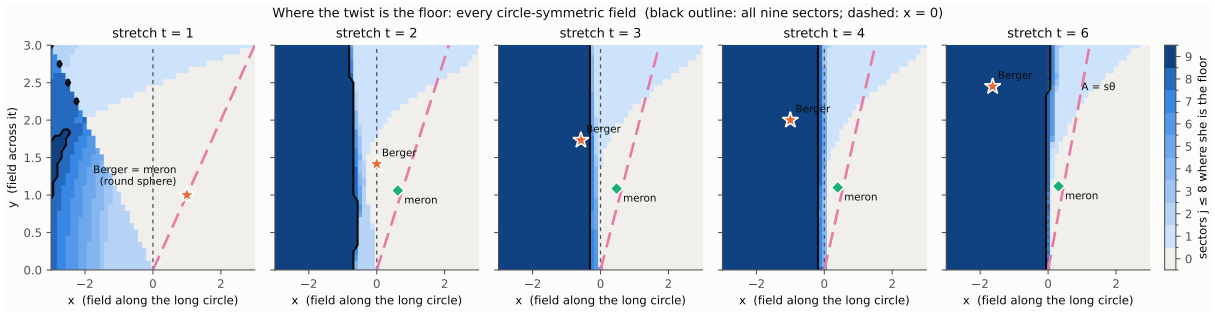

FIGURE 4. Where the twist is the floor, over the circle-symmetric family at $t = 1, 2, 3, 4, 6$ (sectors $j \le 8$; double precision): number of sectors in which the twist is the floor; black outline, all nine; dashed, $x = 0$. Star: the Berger field; diamond: the squashed meron; dashed line: $A = s\theta$. Cells with $x > 0$ counted as all nine are artefacts of stopping at $j = 8$: for $x > 0$ some higher sector undercuts the twist.

with $y \gtrsim 1$ a block that reaches the twist before $B_{4,0}$ does; the companion edge study maps them. Within $j \le 12$ the edge rises with the stretch (for $y = \sqrt{t}$, the $H + GG^*$ edge x^* is -0.68, -0.38, -0.22 at $t = 2, 3, 4$, and the gauge-orthogonal edge -0.66, -0.36, -0.19) and turns positive for strongly stretched spheres with a strong cross field. That is an artefact of the cut-off. *For $x > 0$ the twist is not the floor of $H + GG^*$ in every sector:* in the block of charge $q = l$ the state of weight $m = l$ and type $(1, -1)$ has Rayleigh quotient minus λ equal to $-8lx/\sqrt{t} + O(1)$, negative for large l (`scripts/edge_wp2_symbolic.py`; at $(x, y, t) = (0.1, \sqrt{10}, 10)$ the sector $j = 120$ already falls 0.58 below the twist). The test vector is not gauge-orthogonal, so this is a statement about $H + GG^*$; for the gauge-orthogonal spectrum the same behaviour is seen numerically but not proved by this argument, and the argument does not settle $x = 0$. So the all-sector edge of $H + GG^*$ is at most $\min(x^*_{12}, 0)$. That it equals this is a conjecture for $t \ge 3$; it fails near $t = 2$, where a block of charge $q = l$ undercuts the twist first in the sector $j = 16$ (`research/edge-theorem-20261121/`). The region is bounded below as well: λ grows like x^2, while the gauge-orthogonal state $\hat{\theta}^0 \otimes \tau_0$ of the sector $j = 0$ has a Rayleigh quotient affine in x, so for strongly negative x the twist is not the floor either (`scripts/edge_wp4_design_checks.py`). Along the Berger path, the computed $H + GG^*$ crossing agrees with the algebraic root $t_A = 2.7311222782381144$ of Remark 6.8 to 8×10^{-16}. The gauge-orthogonal crossing is located numerically, at $t^* = 2.698026940521$ to 10^{-12}; its closed form is open. Both crossings occur in the sector $j = 4$; the $H + GG^*$ crossing is through the block $B_{4,0}$. (`scripts/ricci_su2_exact_edge.py`). The thresholds of Remark 6.8 are therefore not special to the Berger field: they are where its path meets the edge. The edge is a crossing, not an avoided one: the twist is alone in its charge block and every competitor found has a different charge, and eigenvalues of different symmetry type may cross where those of one type generically do not [17].

Theorem 7.3 (The floor on a domain)**.** *On $D = \{x \in [-3, -1.5],\ y \in [0.5, 3],\ t \in [3, 10]\}$ the twist is the lowest eigenvalue of $H + GG^*$ in every sector, with multiplicity exactly two, and the lowest gauge-orthogonal eigenvalue there.*

Proof. By the charge blocks, as in Theorem 6.7. *Sectors $j \ge 80$:* the Gershgorin quantity is $G = 4\xi(2l - \xi)(1 - 1/t) + lK_a + E_a -$ (ladder) with $K_a = 4 - 8/t - 4(w_a + x)/\sqrt{t} > 0$ for every non-twist type on D; it is proved positive for $l \ge 40$ on D by interval bisection (`scripts/ricci_su2_family_large_j.py`), and an exact lemma shows that G is, row by row, the diagonal

minus λ minus the exact Gershgorin radius, so $G > 0$ gives the disc bound with no numerical step (`scripts/ricci_su2_gershgorin_lemma_symbolic.py`). *Sectors $j < 80$:* every non-twist block is certified above λ on D by a mean-value interval certificate. On each box the lowest eigenvalue of $B - \lambda I$ at the centre is bounded below by an interval $LDL^{\top}$ certificate with outward rounding and an a posteriori residual, and its variation over the box by $\sum_i \sup\|\partial_i A\|_F$ times the half-widths, the derivatives being enclosed in interval arithmetic. In all, 1,640,935 boxes over the 80 sectors, with no failure and smallest certified per-box slack 1.2×10^{-6} (the slack left on the last box the bisection kept, not a spectral margin) (`scripts/ricci_su2_family_block_cert.py`, receipt `family_block_cert_j80.json`); the few final operations not done in directed rounding carry an error of at most 1.6×10^{-15}, far below that margin (`research/edge-theorem-20261122-proofs/rounding_bound.md`). The two twist blocks are 1×1 with entry λ (Proposition 7.1 and the antiunitary structure, as in Lemma 6.5). The closed-form rows used here are built symbolically by the construction of Lemma 6.5, now with x and y free, and the interval blocks agree with the double-precision closed-form blocks to 1.7×10^{-13} (a check of the interval evaluation). The rows themselves are proved: the Weitzenböck identity and every charge-block row hold exactly for the whole family, x, y, t symbolic (`scripts/ricci_su2_weitzenbock_family_symbolic.py`), and the certificate's own symbolic rows are proved equal to them (`scripts/ricci_su2_rows_transcription_check.py`). □

What does not decide it. Three simpler readings were tested, each written down before its data, and each failed: that the rotation symmetry makes the twist the floor (on the squashed meron and on two $SU(3)$ families with the Maurer–Cartan field it is the floor nowhere); that a field strength aligned with the twist's spin makes it the floor (for $A = s\theta$ with $s > 1$, aligned throughout, it is the floor only of $j = 0$); and that alignment together with the twist alone spanning the lowest eigenspace of R_0 does (on 35 such points of $A = s\theta$ on $(t, 1, 1)$, none). The curvature term alone does not decide it; the comparison above, governed by x, does. The literature checked (the chromomagnetic instability [15, 6], the Weitzenböck formula [2], spectra on Berger spheres [13, 4], homogeneous backgrounds on the round S^3 [5, 10]) did not contain such an edge; the closest is the negative modes of covariantly constant backgrounds on the round sphere.

8 Checks

The round sphere. At $g = (1, 1, 1)$ the Levi-Civita connection is a Yang–Mills field (the meron), and the same construction gives -2, 1, 6, 7 and 10 as the lowest gauge-orthogonal eigenvalues of the sectors $j = 0, \ldots, 4$, in spins 0 to 2. That is the published fluctuation spectrum [10] (`scripts/ricci_su2_round_sphere_check.py`, receipt `research/su2-left-invariant-sector-20261119/round_sphere_check.json`). It has exactly one negative eigenvalue, so index one, as Laquer's theorem requires for a group manifold [12]. Without the commutator it gives the coexact 1-form spectrum of the unit S^3: j^2 and $(j+2)^2$ with their multiplicities, and gauge zeros.

Lattice computations. The two lattice Yang–Mills forces of [3], linearised at the Berger field on triangulations of the Poincaré sphere, resolve the left-invariant sector and two eigenvalues of H in spin 6: $154/3 = 51.33$, the floor, and 59.52, beside the second gauge-orthogonal value 59.50; H itself has lower, gauge-dominated eigenvalues there, and these two are matched by value (Table 2). At $n = 1, 2$ every left-invariant mode was identified by its eigenvector, as the lattice

Table 2. Exact eigenvalues of H at the Berger field against the real part of the lattice eigenvalue [3]. ^{g}: pure gauge; ^{m}: a mixture. At $n = 1, 2$ the spin-0 entries are identified by eigenvector; at $n = 3$, and for spin 6, by value.

Exact	mult.	native $n = 1$	native $n = 2$	native $n = 3$	two-ring $n = 3$
spin 0					
$-14/3 = -4.6667$	2	-4.475	-4.651	-4.661	-4.651
$(29 - \sqrt{1345})/3 = -2.5581^{m}$	2	-2.191	-2.504	-2.529	-2.529
$16 - 2\sqrt{73} = -1.0880$	1	$-0.674 \pm 0.80i$	-1.041	-1.088	-1.084
8^{g}	1	7.083	7.869	7.954	7.931
$(29 + \sqrt{1345})/3 = 21.891^{m}$	2	20.71	21.77	21.86	21.80
$16 + 2\sqrt{73} = 33.088$	1	32.64	33.44	33.29	33.16
spin 6, *full Hessian*					
51.33	2	—	53.41	52.27	49.27
59.52	2	—	61.92	60.69	57.50

eigenvector with the largest overlap with the nine discrete left-invariant directions (overlap 0.76 to 0.999). The mode $16 - 2\sqrt{73}$ is split by the non-symmetric lattice Jacobian, between a real eigenvalue and a complex pair. The lattice linearises the force in link coordinates at a field that is not at rest, where its Jacobian differs from the Hessian by a term proportional to the gradient; so it is compared with the full Hessian H, not with the slice.

9 Discussion

Scope. The result concerns the classical linearised operator. At a field that is not at rest, how gauge directions are separated from physical ones is a choice; the theorem is stated for the L^2 (Coulomb) slice, and the full Hessian is given alongside. The theorem is not a statement about the quantum theory and not a Yang–Mills mass gap.

What is special. The value $(j(j + 2) - 14)/3$ is tied to $t = 3$, but the mechanism is not. The circular eigenvector uses only the rotation symmetry about the long axis and the top weight, so it is an exact eigenvector at every squashing (Proposition 6.1), with the stretch entering the formula rationally, and it is a lowest Landau level at every squashing (Proposition 6.2). Whether it is the *floor* depends on t, and on the background: symmetry alone does not make it the floor (Section 6, "Further observations"). It is the floor throughout $[2.75, 10]$ in every sector, and (numerically) not the floor of every sector below $t^* \approx 2.698$, where the sector $j = 4$ is the first to have a lower eigenvalue. At the round metric and at $(2, 1, 1)$ other modes are lower. The relation to the round Poincaré sphere's first scalar eigenvalue, $168/3$, is special to $t = 3$, the unique stretch at which the lift of the twist depends only on the Casimir (Remark 6.10); in general the floor gap is $(240 - 24t)/t$.

The landscape and the flow. The Berger field is not at rest, and under the Yang–Mills flow it slides towards a flat field faster than any twist grows (Section 6). So the floor law is a statement about the second variation at the Berger field, the shape of the landscape there. Its non-flat resting relative, the squashed meron, has, at every stretch sampled, one unstable direction in the left-invariant sector and none in the higher sectors examined ($1 \le j \le 12$), and no twist goes soft

on it there.

Open questions. The edge $x^*(t, y)$ as a theorem in every sector, and its other competitors in closed form; The floor outside $[2.75, 10]$; the gauge-orthogonal threshold t^* in closed form (that of $H + GG^*$ is algebraic, Remark 6.8); the floor along the Ricci flow of the metric; Yang–Mills fields that are not left-invariant on the squashed spheres, where a twist might go soft at rest; and the higher eigenvalues of each sector.

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Code and data availability

All programs and certificates are in the GIGI repository, `https://github.com/nurdymuny/gigi` (tag `floor-theorem-v3.3`; the scripts are on that tag, not yet on the main branch), with their outputs in `research/su2-left-invariant-sector-20261119/`:

- the left-invariant sector: `scripts/ricci_su2_left_invariant_sector.py`;
- the sector matrices: `scripts/ricci_su2_spin_sector.py`;
- gauge content: `scripts/ricci_su2_sector_gauge_check.py`;
- Proposition 4.1: `scripts/ricci_su2_sector_formula.py`;
- the spin-0 slice and the double-precision range: `scripts/ricci_su2_sector_minimality.py`;
- Lemma 5.1 and the certificates: `scripts/ricci_su2_gap_certificates.py`.

For Section 6, with outputs in `research/su2-floor-family-20261119/`:

- Proposition 6.1: `scripts/ricci_su2_sector_formula_squashed.py`;
- Theorem 6.4: `scripts/ricci_su2_gap_certificates_t.py` (any rational t);
- Theorem 6.7: `scripts/ricci_su2_floor_interval_cert.py` (fifteen sectors), `scripts/ricci_su2_block_interval_cert.py` ($j \le 99$), `scripts/ricci_su2_floor_large_j.py` ($j \ge 100$);
- Proposition 6.2: `scripts/ricci_su2_weitzenbock_symbolic_t.py`, `scripts/ricci_su2_landau_symbolic.py`;
- Lemma 6.5: `scripts/ricci_su2_charge_selection_symbolic.py`;
- the charge blocks: `scripts/ricci_su2_charge_block_direct.py`; their closed-form rows: `scripts/ricci_su2_block_rows_symbolic.py`;
- Remark 6.8: `scripts/ricci_su2_floor_family_scan.py`, `scripts/ricci_su2_tstar_exact.py`, `scripts/ricci_su2_tA_scan.py`;
- the flow experiment: `scripts/ricci_su2_soft_mode_flow.py`;
- Section 7: `scripts/ricci_su2_twist_family_closed_form.py`, `scripts/ricci_su2_exact_edge.py`,

`scripts/ricci_su2_edge_polynomial.py`, `scripts/ricci_su2_twist_family_map.py`, `scripts/ricci_su2_family_large_j.py`, `scripts/ricci_su2_family_block_cert.py`;

- the outward error budget of every certificate: `scripts/ricci_su2_outward.py`, with its adversarial tests in `scripts/ricci_su2_outward_test.py`;
- the resting relative's index for $t < 1$: `scripts/ricci_su2_meron_index_below1.py`;
- symmetry does not make the floor: `scripts/ricci_su2_test2_family.py`, `scripts/ricci_su2_gyromagnetic_split.py`, `scripts/ricci_su2_meron_ceiling_check.py`;
- resting fields: `scripts/ricci_su2_resting_fields.py`, `scripts/ricci_su2_resting_fields_general.py`, `scripts/ricci_su2_meron_family_soft.py`.

This paper, its source, and the code and data above are archived on Zenodo under the concept DOI `https://doi.org/10.5281/zenodo.24386613`, which resolves to the latest version.

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